AN EASY DETERMINATION OF THE FIBONACCI AND PELL SQUARES

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Abstract. We present a short derivation of the known perfect squares in the Fibonacci and Pell sequences, using an elementary approach leading to several Thue equations which are then solved using the online Magma calculator.

1. Introduction

The Fibonacci sequences $\{F_n\}$ and the Pell sequence $\{P_n\}$ are special cases of the Lucas sequence $\{U_n\} = \{U_n(P, Q)\}$ defined by:

$$U_0 = 0,\ U_1 = 1,\ U_{n+1} = PU_n - QU_{n-1} \quad (n \geq 1). \tag{1.1}$$

Thus, $\{F_n\} = \{U_n(1, -1)\}$ and $\{P_n\} = \{U_n(2, -1)\}$.

Interest in the square terms in Lucas sequences was evident as long ago as 1962 when Ogilvy [12, p. 100] raised the question concerning square terms in the Fibonacci sequence. As is well-known, the Fibonacci squares were subsequently found, using elementary arguments, by Wyler [15] and Cohn [2] in 1964, and the Pell squares were found by Ljunggren [6] in 1942, using algebraic number theory and p-adic analysis.

McDaniel and Ribenboim announced in 1992 [8], that they had determined the square terms in $\{U_n(P, Q)\}$ for all odd relatively prime P and Q using elementary means, and the proof appeared in print in 1996 [14].

It appears increasingly unlikely that the methods of elementary number theory will be successfully employed in finding the square terms in Lucas sequences having an *even* parameter. During the last two decades the square terms have been found, for P even, only for $\{U_n(P, -1)\}$ and $\{U_n(P, 1)\}$. Kagawa and Terai [4] found the squares in $\{U_n(P, -1)\}$, for certain restricted values of P using properties of elliptic curves, Nakamula and Pethő [10] found the squares, for all P in $\{U_n(P, -1)\}$, by first transforming certain equations to find units in orders of infinite families of quartic algebraic number fields, and Mignotte and Pethő [11] used the same approach in finding the squares in $\{U_n(P, 1)\}$. We discuss these families of Lucas sequences more fully in the Discussion Section.

In this paper, we present a very simple method for obtaining the square terms of $\{F_n\}$ and $\{P_n\}$ for n odd and are then able to immediately find the square terms for n even. The approach we use is quite general and, when applied to the sequences $\{U_n(P, \pm 1)\}$, provides a nice concise alternate determination of the squares for the two sequences. The approach, *in general*, when applied to a family of Lucas sequences, leads to a parameterized Thue equation which may then be solved using one of the powerful methods which have been developed in recent years for solving Thue equations. Our approach then is *hybrid* in the sense that elementary means are used for obtaining a parameterized Thue equation, and more advanced means are used for solving it.

When this approach is applied to a *single* Lucas sequence such as $\{F_n\}$ or $\{P_n\}$, as in this paper, the resulting Thue equations may be solved easily: just as in an earlier time number theorists were spared tedious calculations through the use of tables such as J. W. L. Glaisher's "Number Divisor Tables" [3] so today online calculator tools are readily available to do the calculating for us. We shall use the online Magma calculator to solve four Thue equations. (Running time is less than one second in each case.)

It is of interest that the squares in the Fibonacci and Pell sequences (and, in fact, any Lucas sequence for which P is given and $Q = \pm 1$) may thus be found by the interested student using neither congruences nor the quadratic character of the terms of the sequences, but using instead as the principal tools, the Pythagorean Theorem and the Magma program. Our proof is quite understandable to any student who has had high school algebra.

Theorem 1.1. *The only perfect squares in the Fibonacci sequence are 0, 1, and 144.*

Theorem 1.2. *The only perfect squares in the Pell sequence are 0, 1, and 169.*

We shall refer to a term which is 2 times a square as a "double-square", and also prove the following theorems.

Theorem 1.3. *The only double-squares in the Fibonacci sequence are* $F_3 = 2$ *and* $F_6 = 8$.

Theorem 1.4. *The only double-square in the Pell sequence is* $P_2 = 2$.

2. Some Identities, Properties and Lemmas

For all non-negative integers n:

(1) $U_{n+1}^2 - QU_n^2 = U_{2n+1}$.
(2) $U_{n+1}U_{n-1} - U_n^2 = -Q^{n-1}$.
(3) For even P, U_n is odd if and only if n is odd.

These identities/properties are well-known and listed in [13].

Lemma 2.1. *If m is an odd integer, then, for $P = 1$ or 2, $U_m(P, -1)$ is a square if and only if $(P, m) = (1, 1),\ (2, 1)$ or $(2, 7)$.*

Proof. We first note that $U_m(P, -1)$ is a square if $(P, m) = (1, 1),\ (2, 1)$ or $(2, 7)$. For the necessity let $m = 2n+1$ and assume $U_{2n+1} = z^2$ for some integer z. Then from (1), $U_n^2 + U_{n+1}^2 = z^2$. This Pythagorean equation is primitive since $\gcd(U_n, U_{n+1}) = 1$. It follows that, for relatively prime positive integers a and b, of opposite parity, either

(i) $U_n = 2ab$, $U_{n+1} = a^2 - b^2$, and $z = a^2 + b^2$ or
(ii) $U_n = a^2 - b^2$, $U_{n+1} = 2ab$, and $z = a^2 + b^2$.

If (i), then from (1.1), $U_{n-1} = U_{n+1} - PU_n = a^2 - b^2 - 2Pab$. Substituting these values in (2), we have

$$(a^2 - b^2)(a^2 - b^2 - 2Pab) - 4a^2b^2 = (-1)^n.$$

And upon simplifying, we obtain the Thue equation

$$a^4 - 2Pa^3b - 6a^2b^2 + 2Pab^3 + b^4 = (-1)^n.$$

Using the online Magma calculator, we solve this Thue equation for $P = 1$ and $P = 2$, finding that there is no solution with $a > b > 0$ when $P = 1$, and for $P = 2$ the only positive solution is $(3, 2)$. However, $(a, b) = (3, 2)$ implies that $U_n = 2ab > U_{n+1} = a^2 - b^2$. Hence, (i) is impossible.

If (ii), then from (1.1), $U_{n-1} = U_{n+1} - PU_n = 2ab - P(a^2 - b^2)$. Substituting these values in (2) we have

$$2ab[(2ab - P(a^2 - b^2)] - (a^2 - b^2)^2 = (-1)^n,$$

which upon simplifying gives us

$$a^4 + 2Pa^3b - 6a^2b^2 - 2Pab^3 + b^4 = (-1)^n.$$

Solving for $P = 1$, we again get no solution, and for $P = 2$, the only positive solution is $(3, 2)$, which yields $U_{2n+1}(2, -1) = z^2 = 169 = U_7$. Hence, there are no additional solutions. □

Lemma 2.2. *If m is an odd integer, then for $P = 1$ or 2, $2U_m(P, -1)$ is a square if and only if $(P, m) = (1, 3)$.*

Proof. We note that when m is odd, $2U_m(2, -1)$ is not a square since by (3), $2U_m(2, -1)$ is an even number not divisible by 4. Thus, we may assume $P = 1$, that is, $U_m(P, -1) = F_m$.

If $(P, m) = (1, 3)$, $2U_m(P, -1) = 2 \cdot F_3 = 4$ is a square.

Let $m = 2n + 1$, $n \geq 0$, and assume that $F_{2n+1} = 2z^2$ for some integer z. Then from (1), $F_n^2 + F_{n+1}^2 = 2z^2$, which may be rewritten as the Pythagorean equation.

$$[(F_{n+1} + F_n)/2]^2 + [(F_{n+1} - F_n)/2]^2 = z^2. \tag{2.1}$$

It is readily seen that F_n, F_{n+1} and z are all odd and the three terms of (2.1) are relatively prime in pairs. Hence, for positive, relatively prime integers a and b of opposite parity, with $a > b$, either

(i) $(F_{n+1} - F_n)/2 = 2ab$, $(F_{n+1} + F_n)/2 = a^2 - b^2$ and $z = a^2 + b^2$, or
(ii) $(F_{n+1} - F_n)/2 = a^2 - b^2$, $(F_{n+1} + F_n)/2 = 2ab$ and $z = z^2 + b^2$.

If (i), we find that $F_n = a^2 - b^2 - 2ab$, $F_{n+1} = a^2 - b^2 + 2ab$, and using (1.1), $F_{n-1} = 4ab$. Substituting these values in (2), we have

$$(a^2 - b^2 + 2ab)(4ab) - (a^2 - b^2 - 2ab)^2 = (-1)^n,$$

and upon simplifying,

$$a^4 - 8a^3b - 6a^2b^2 + 8ab^3 + b^4 = (-1)^{n+1}.$$

If (ii), we obtain, similarly, the Thue equation

$$a^4 + 8a^3b - 6a^2b^2 - 8ab^3 + b^4 = (-1)^n.$$

Solving these Thue equations using the online Magma calculator we find for each equation that there is no solution (a, b) for which $a > b > 0$. □

We require in the next lemma, the companion Lucas sequence $\{V_n\} = \{V_n(P, Q)\}$ which satisfies the same recurrence relation as $\{U_n\}$ but with initial terms $V_0 = 2$, and $V_1 = P$.

Lemma 2.3. *Let $n = 2^k m$, $k \geq 1$ and m odd. Then U_n is a square or twice a square only if U_m and each of V_{jm} for $1 \leq j \leq 2^{k-1}$ is a square or 2 times a square.*

Proof. From the well-known identity $U_n = U_{n/2}V_{n/2}$ with $n = 2^k m$, we easily obtain $U_n = U_m V_m V_{2m} V_{4m} \cdots V_{tm}$, for $t = 2^{k-1}$. The gcd of the factors on the right-hand side is either 1 or 2, (see [7]) and therefore U_n is a square or twice a square only if each of the factors U_m and V_{jm} for $1 \leq j \leq 2^{k-1}$ is a square or 2 times a square. □

3. Proof of the Theorems

We shall use the standard notations $\{L_n\}$ and $\{R_n\}$ for the companion sequences of the Fibonacci and Pell sequences, respectively.

Simultaneous Proof of Theorems 1.1 and 1.3. By Lemma 2.1, $F_n = F_1 = 1$ is the only square for n odd, and by Lemma 2.2, $F_3 = 2$ the only odd-indexed term of the Fibonacci sequence which is 2 times a square. By Lemma 2.3, then, for n even, F_n is a square or 2 times a square only if $n = 2^k$ or $n = 2^k \cdot 3$.

If $m = 1$, we observe that $L_{2m} = L_2 = 3$ is neither a square nor 2 times a square. By Lemma 2.3 then, F_n is not a square nor a double-square for $n = 2^k$ if $k > 1$. If $m = 3$, $L_{4m} = L_{12} = 322$. Since 322 is neither a square nor 2 times a square F_n is not a square nor a double-square for $n = 2^k \cdot 3$ if $k > 2$. One readily finds for the values of k not excluded (that is, for $n \leq 12$) that the Fibonacci squares are $F_0 = 0$, $F_1 = 1$, $F_2 = 1$, and $F_{12} = 144$, and the double squares are $F_3 = 2$ and $F_6 = 8$. □

Simultaneous Proof of Theorem 1.2 and 1.4. By Lemma 2.1, $P_1 = 1$ and $P_7 = 169$ are the only odd-indexed terms of the Pell sequence which are squares and by Lemma 2.2, there are no such terms which are 2 times a square. By Lemma 2.3, then, the even-indexed square terms of the Pell sequence must be of the form P_n with $n = 2^k$ or $n = 2^k \cdot 7$.

If $m = 1$, $R_2 = 6$ is neither a square nor 2 times a square. By Lemma 2.3, then, P_n is not a square nor a double-square for $n = 2^k$ if $k > 1$. If $m = 7$, $R_7 = 29$; since 29 is neither a square nor 2 times a square, P_n is not a square nor a double-square for $n = 2^k \cdot 7$ if $k \geq 1$. Again, one readily finds for the values of k not excluded (that is, for $n \leq 7$) that the Pell squares are $P_0 = 0$, $P_1 = 1$, $P_7 = 169$, and the only double square is $P_2 = 2$. □

4. Discussion

The papers [4], [10], and [11] all contain results in addition to those pertaining to the squares in the respective sequences. For those results having to do with the square terms, however, the approach of this paper is considerably more efficient. This is particularly true of [4] and [10]; the Thue equations found in our Lemmas 2.1 and 2.2 are special cases of the Thue equation needed for finding the odd-indexed terms in $\{U_n(P, -1)\}$, and this Thue equation was solved in an earlier paper by G. Lettl and A. Pethő [5], and independently by Chen and Voutier [1]; the even-indexed square terms are then readily found from the odd-indexed square terms using our Lemma 2.3 (with the help of a result of Ljunggren's). Whether this approach to obtaining the Thue equations for other families of Lucas sequences, or other related problems, will be fruitful in obtaining the square terms remains to be seen; work along these lines is under way.

For the benefit of those readers who are not familiar with the MAGMA program, it is available online at `http://magma.maths.usyd.edu.au/calc/` and MAGMA's online help page occurs at `http://magma.maths.usyd.edu.au/magma/handbook/text/453#3670`.

Here's an example illustrating its use: find all integer solutions to the quartic Thue equation $x^4 - 2x^3y - 6x^2y^2 + 2xy^3 + y^4 = -4$. To solve this equation, copy the following into the MAGMA calculator and then click on "Submit" at the bottom of the calculator:

```
R<x> := PolynomialRing(Integers());
f := x^4 - 2*x^3 - 6*x^2 + 2*x + 1;
T := Thue(f);
Solutions(T,-4);
```

5. Commentary

It is well-known that any determination of the perfect squares in the Pell sequence provides an alternative proof that the solutions of the Diophantine equation $x^2 + 1 = 2y^4$ are $(x, y) = (1, 1)$ and $(x, y) = (239, 13)$. L. J. Mordell asked in [9] whether a simpler proof than that given by Ljunggren [6] of the solutions of this equation might not be available. While one might be inclined to suggest that the proof herein qualifies, I would hasten to say that it does not. The proof is deceptively simple, for the mathematics required to construct the online Magma calculator involves technically difficult concepts.

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