FIBONACCI EXPRESSIONS ARISING FROM A COIN-TOSSING SCENARIO INVOLVING PAIRS OF CONSECUTIVE HEADS

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Abstract. In this article we study a combinatorial scenario which generalizes the well-known problem of enumerating sequences of coin tosses containing no consecutive heads. It is shown how to derive formulas enumerating, for a fixed value of k, the sequences of length n containing exactly k distinct pairs of consecutive heads. We obtain both exact expressions and asymptotic relations.

1. Introduction

Suppose that a coin is tossed 12 times and the resulting sequence of heads and tails is recorded. Two possible outcomes are as follows:

(a) HTTTHTHTTTTH and (b) HTTHHTTHHHHT.

Notice that sequence (a) contains no consecutive heads whereas (b) does. It is in fact well-known [3, 7] that of the 2^n possible outcomes when a coin is tossed n times, the number containing no consecutive heads is given by F_{n+2}. Looking at sequence (b) again, we can actually be a little more specific and say that it contains exactly 4 distinct pairs of consecutive heads. Counting from left to right, these pairs are positioned at $(4, 5)$, $(8, 9)$, $(9, 10)$ and $(10, 11)$. Thus, on using $U(n, k)$ to denote the number of sequences of length n containing exactly k HH's, we see that (a) contributes 1 to $U(12, 0)$ while (b) contributes 1 to $U(12, 4)$.

Our aim here is to derive formulas that allow us to enumerate, for a fixed value of k, the sequences of length n containing exactly k distinct pairs of consecutive heads. In an interesting though rather incomplete initial foray into this problem [2], the following recurrence relation was obtained for $U(n, k)$:

$$U(n, k) = U(n-1, k) + U(n-1, k-1) + U(n-2, k) - U(n-2, k-1), \tag{1.1}$$

where $n \geq 2$, $k \geq 0$, $U(m, -1) = 0$ for all $m \geq 0$, and $U(0, 0) = 1$ by definition. In order to use (1.1) to calculate $U(n, k)$ recursively, we need simply to note that $U(1, 0) = 2$ and $U(0, m) = U(1, m) = 0$ for all $m \geq 1$.

As is shown in [2], it is reasonably straightforward to derive the relation (1.1) by using the intermediate results

$$UT(n, k) = UT(n-1, k) + UH(n-1, k) \tag{1.2}$$

and

$$UH(n, k) = UT(n-1, k) + UH(n-1, k-1), \tag{1.3}$$

where $UT(n, k)$ and $UH(n, k)$ represent the number of sequences of n tosses containing exactly k HH's such that the last toss is a tail and a head, respectively. Result (1.2) is true since any sequence A of length $n-1$ containing exactly k HH's either ends in a head or a tail, and the sequence of length n that results when a tail is appended to the end of A still has exactly this many pairs of HH's. On the other hand, a sequence of length n with exactly k HH's such that the last toss is a head either has its first $n-1$ tosses containing exactly k HH's and ending a tail or containing exactly $k-1$ HH's and ending in a head.

In order to obtain explicit formulas for $U(n,k)$ from (1.1), we illustrate a nice application of exponential generating functions. It is shown that $U(n,k)$ becomes an ever-more complex expression in n involving the Fibonacci numbers as k increases. A general asymptotic relation for $U(n,k)$ is also obtained.

2. Some Results on Exponential Generating Functions

The *exponential generating function* $G(x)$ for the sequence $a_0, a_1, a_2, \ldots$ is defined as

$$\begin{aligned} G(x) &= \frac{a_0}{0!} + \frac{a_1}{1!}x + \frac{a_2}{2!}x^2 + \frac{a_3}{3!}x^3 + \cdots \\ &= \sum_{k=0}^{\infty} \frac{a_k}{k!}x^k, \end{aligned}$$

where the coefficient of $x^n/n!$ in this series is a_n [1, 4]. Since

$$\begin{aligned} G'(x) &= \frac{a_1}{0!} + \frac{a_2}{1!}x + \frac{a_3}{2!}x^2 + \cdots \\ &= \sum_{k=0}^{\infty} \frac{a_{k+1}}{k!}x^k, \end{aligned} \tag{2.1}$$

a_{n+1} is the coefficient of $x^n/n!$ in $G'(x)$. Also,

$$\begin{aligned} xG(x) &= \frac{a_0}{0!}x + \frac{a_1}{1!}x^2 + \frac{a_2}{2!}x^3 + \frac{a_3}{3!}x^4 + \cdots \\ &= \frac{a_0}{1!}x + \frac{2a_1}{2!}x^2 + \frac{3a_2}{3!}x^3 + \frac{4a_3}{4!}x^4 + \cdots \\ &= \sum_{k=1}^{\infty} \frac{ka_{k-1}}{k!}x^k, \end{aligned} \tag{2.2}$$

so the coefficient of $x^n/n!$ in $xG(x)$ is na_{n-1}. The above may be generalized to show that the coefficient of $x^n/n!$ in $x^mG(x)$ is $n(n-1)\cdots(n-m+1)a_{n-m}$.

In particular, the exponential generating function for the Fibonacci numbers is given by

$$H(x) = \frac{2}{\sqrt{5}} \exp\left(\frac{x}{2}\right) \sinh\left(\frac{x\sqrt{5}}{2}\right) \tag{2.3}$$

(see [5] or sequence A000045 in [6], for example). This function will play a key role in our quest for formulas enumerating the sequences of length n containing a certain fixed number of pairs of consecutive heads.

3. Initial Calculations

The exponential generating function of $U(n,k)$ for fixed k is given by

$$G_k(x) = \sum_{n=0}^{\infty} \frac{U(n,k)}{n!}x^n. \tag{3.1}$$

On using (1.1) and setting $k = 1$ we obtain

$$\sum_{n=0}^{\infty} \frac{U(n+2,1)}{n!} x^n = \sum_{n=0}^{\infty} \frac{U(n+1,1)}{n!} x^n + \sum_{n=0}^{\infty} \frac{U(n+1,0)}{n!} x^n + \sum_{n=0}^{\infty} \frac{U(n,1)}{n!} x^n - \sum_{n=0}^{\infty} \frac{U(n,0)}{n!} x^n,$$

from which it follows, by way of (2.1), that

$$G_1''(x) - G_1'(x) - G_1(x) = G_0'(x) - G_0(x). \tag{3.2}$$

However, since $U(n,0) = F_{n+2}$, it is the case, utilizing (2.1) once more, that $G_0(x) = H''(x)$. Thus (3.2) gives the linear second-order differential equation

$$\begin{aligned} G_1''(x) - G_1'(x) - G_1(x) &= H'''(x) - H''(x) \\ &= \frac{1}{5} \exp\left(\frac{x}{2}\right) \left\{ 5 \cosh\left(\frac{x\sqrt{5}}{2}\right) + \sqrt{5} \sinh\left(\frac{x\sqrt{5}}{2}\right) \right\}. \end{aligned} \tag{3.3}$$

This has the auxiliary equation $\lambda^2 - \lambda - 1 = 0$ with solutions

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \lambda_2 = \frac{1-\sqrt{5}}{2}.$$

The complementary function is therefore given by

$$\begin{aligned} &A \exp\left(\frac{1+\sqrt{5}}{2} x\right) + B \exp\left(\frac{1-\sqrt{5}}{2} x\right) \\ &\qquad = \exp\left(\frac{x}{2}\right) \left\{ (A+B) \cosh\left(\frac{x\sqrt{5}}{2}\right) + (A-B) \sinh\left(\frac{x\sqrt{5}}{2}\right) \right\}, \end{aligned}$$

where $A, B \in \mathbb{R}$. Then, as

$$\exp\left(\frac{x}{2}\right) \cosh\left(\frac{x\sqrt{5}}{2}\right) \qquad \text{and} \qquad \exp\left(\frac{x}{2}\right) \sinh\left(\frac{x\sqrt{5}}{2}\right)$$

are part of the complementary function, we try the particular integral

$$Cx \exp\left(\frac{x}{2}\right) \cosh\left(\frac{x\sqrt{5}}{2}\right) + Dx \exp\left(\frac{x}{2}\right) \sinh\left(\frac{x\sqrt{5}}{2}\right)$$

for some $C, D \in \mathbb{R}$.

On using the initial conditions $G_1(0) = G_1'(0) = 0$, the solution to (3.3) is found to be

$$G_1(x) = \frac{1}{25} \exp\left(\frac{x}{2}\right) \left\{ 5x \cosh\left(\frac{x\sqrt{5}}{2}\right) + \sqrt{5}(5x-2) \sinh\left(\frac{x\sqrt{5}}{2}\right) \right\}. \tag{3.4}$$

In order to obtain a formula for $U(n,1)$ it is now a matter of extracting the coefficient of $x^n/n!$ from the right-hand side of (3.4). First, from (2.3) we obtain

$$H'(x) = \exp\left(\frac{x}{2}\right) \left\{ \cosh\left(\frac{x\sqrt{5}}{2}\right) + \frac{1}{\sqrt{5}} \sinh\left(\frac{x\sqrt{5}}{2}\right) \right\}. \tag{3.5}$$

Then, on using (3.5) and (2.3) in turn, we have

$$\exp\left(\frac{x}{2}\right)\cosh\left(\frac{x\sqrt{5}}{2}\right) = H'(x) - \frac{1}{\sqrt{5}}\exp\left(\frac{x}{2}\right)\sinh\left(\frac{x\sqrt{5}}{2}\right)$$
$$= H'(x) - \frac{1}{2}H(x). \quad (3.6)$$

Therefore, from (3.4), (3.6), and (2.3), it is the case that

$$G_1(x) = \frac{x}{5}\left(H'(x) - \frac{1}{2}H(x)\right) + \frac{5x-2}{10}H(x)$$
$$= \frac{2x}{5}H(x) - \frac{1}{5}H(x) + \frac{x}{5}H'(x).$$

Using this result, along with (2.1), (2.2), and (3.1), we obtain $U(n,1)$, the coefficient of $x^n/n!$ in $G_1(x)$:

$$U(n,1) = \frac{2n}{5}F_{n-1} - \frac{1}{5}F_n + \frac{n}{5}F_n$$
$$= \frac{1}{5}\left\{n\left(F_{n+1} + F_{n-1}\right) - F_n\right\}, \quad (3.7)$$

which is valid for any $n \geq 0$.

4. Further Results

This process may now be continued indefinitely. Next, we have

$$G_2''(x) - G_2'(x) - G_2(x) = G_1'(x) - G_1(x)$$
$$= \frac{2}{25}\exp\left(\frac{x}{2}\right)\left\{5x\cosh\left(\frac{x\sqrt{5}}{2}\right) + 3\sqrt{5}\sinh\left(\frac{x\sqrt{5}}{2}\right)\right\}. \quad (4.1)$$

The solution of (4.1) is

$$G_2(x) = \frac{1}{125}\exp\left(\frac{x}{2}\right)\left\{20x\cosh\left(\frac{x\sqrt{5}}{2}\right) + \sqrt{5}(5x^2-8)\sinh\left(\frac{x\sqrt{5}}{2}\right)\right\}. \quad (4.2)$$

Such differential equations are somewhat tedious to solve by hand, and we performed the calculations with the assistance of *Mathematica* [8]. In conjunction with (3.1), (2.1), (2.2), and the generalization of (2.2), (4.2) gives,

$$U(n,2) = \frac{1}{50}\left\{n(5n-1)F_{n-2} + 4(n-2)F_n\right\}$$

for $n \geq 0$, where we adopt the convention that $F_{-m} = (-1)^{m+1}F_m$ for any $m \geq 0$.

Similarly we have

$$G_3''(x) - {G'}_3(x) - G_3(x)$$
$$= G_2'(x) - G_2(x)$$
$$= \frac{1}{250}\exp\left(\frac{x}{2}\right)\left\{5x(5x-4)\cosh\left(\frac{x\sqrt{5}}{2}\right) + \sqrt{5}\left(-5x^2+40x+8\right)\sinh\left(\frac{x\sqrt{5}}{2}\right)\right\}.$$

This has the solution

$$\frac{1}{750}\exp\left(\frac{x}{2}\right)\left\{5x(-x^2+9x+6)\cosh\left(\frac{x\sqrt{5}}{2}\right)+\sqrt{5}\left(5x^3-3x^2-18x-12\right)\sinh\left(\frac{x\sqrt{5}}{2}\right)\right\},$$

from which we obtain

$$U(n,3)=\frac{1}{150}\left\{n(n-1)(n-2)\left(F_{n-3}+F_{n-5}\right)+3n\left((n-1)F_{n-1}+2(n-2)F_{n-3}\right)-6F_n\right\},$$

and so on.

Each member of the resulting family of linear second-order differential equations has the form

$$\begin{aligned} G_k''(x)-G_k'(x)-G_k(x) &= G_{k-1}'(x)-G_{k-1}(x) \\ &= a\exp\left(\frac{x}{2}\right)\left\{f_k(x)\cosh\left(\frac{x\sqrt{5}}{2}\right)+g_k(x)\sqrt{5}\sinh\left(\frac{x\sqrt{5}}{2}\right)\right\}, \quad (4.3)\end{aligned}$$

for some $a\in\mathbb{Q}$ and polynomials $f_k(x)$ and $g_k(x)$ having integer coefficients and, for $k\geq 3$, degree $k-1$. The solution to (4.3) is of the form

$$G_k(x)=b\exp\left(\frac{x}{2}\right)\left\{r_k(x)\cosh\left(\frac{x\sqrt{5}}{2}\right)+s_k(x)\sqrt{5}\sinh\left(\frac{x\sqrt{5}}{2}\right)\right\}, \quad (4.4)$$

where $b\in\mathbb{Q}$ and $r_k(x)$ and $s_k(x)$ are polynomials having integer coefficients and, for $k\geq 3$, degree k.

From this it is possible to deduce that, for any particular value of k, $U(n,k)$ can be expressed as

$$\sum_{i=0}^{m} h_i(n)F_{n-i},$$

for some integer $m\geq 0$ and family $\{h_i(n): i=0,1,\ldots,m\}$ of polynomials in n (some of which could be the zero polynomial). It needs to be born in mind of course that a particular representation of $U(n,k)$ is certainly not unique; indeed, the Fibonacci recurrence relation allows us to express these representations in different ways.

5. Asymptotic Results

First, on using the result

$$F_n\sim\frac{\phi^n}{\sqrt{5}},$$

which can be found in [4], for example, (3.7) gives

$$\begin{aligned} U(n,1) &= \frac{1}{5}\left\{n\left(F_{n+1}+F_{n-1}\right)-F_n\right\} \\ &\sim \frac{n}{5}\left(F_{n+1}+F_{n-1}\right) \\ &\sim \frac{n}{5}\left(\frac{\phi^{n+1}}{\sqrt{5}}+\frac{\phi^{n-1}}{\sqrt{5}}\right) \\ &= \frac{n\phi^{n-1}}{5\sqrt{5}}(2+\phi).\end{aligned}$$

Then similar, if somewhat more involved, calculations yield

$$U(n,2) \sim \frac{n^2\phi^{n-2}}{10\sqrt{5}} \quad \text{and} \quad U(n,3) \sim \frac{n^3\phi^{n-3}}{150\sqrt{5}}(3-\phi).$$

In general, it is a question of picking out the dominant terms in the expression (4.4) for $G_k(x)$ (and hence for $U(n,k)$), which, for $k \geq 3$, arise as the coefficients of x^k in both $r_k(x)$ and $s_k(x)$. It may be shown from this that

$$U(n,k) \sim \frac{n^k\phi^{n-k}}{k!5^{\frac{k+1}{2}}}\left(F_{k-1} - \phi F_{k-2}\right) \quad \text{and} \quad U(n,k) \sim \frac{n^k\phi^{n-k}}{k!5^{\frac{k+2}{2}}}\left(L_{k-1} - \phi L_{k-2}\right)$$

for k even and k odd, respectively, and thus that

$$U(n,k) \sim \frac{n^k\phi^{n-k}}{k!5^{\frac{k+1}{2}}}(-1)^k\left(F_{k-1} - \phi F_{k-2}\right).$$

Then, since

$$(-1)^k\left(\frac{F_{k-1} - \phi F_{k-2}}{\phi^2 5^{\frac{k}{2}}}\right) = \frac{1}{(2+\phi)^k},$$

we have the result

$$U(n,k) \sim \frac{n^k\phi^{n-k+2}}{k!\sqrt{5}(2+\phi)^k} \tag{5.1}$$

for any fixed $k \geq 0$. The asymptotic formula (5.1) does, however, give rather poor approximations for small values of n. For example, we have to wait until $n = 186$ before it provides us with an approximation for $U(n,2)$ having a relative error of less than 1%.

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