QUADRATIC IDENTITIES FOR A CLASS OF FIBONACCI-LIKE POLYNOMIALS

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Abstract. We find and prove quadratic relations for polynomial analogues of Fibonacci numbers introduced by K. Dilcher and K. Stolarsky.

1. Introduction

Many classical integer sequences have their natural polynomial analogues. For instance, various analogues of the Fibonacci numbers were studied by L. Carlitz [1, 2] and J. Cigler [3, 4]. Recently, K. Dilcher and K. B. Stolarsky [5] introduced two new polynomial counterparts of the Fibonacci sequence. Originally these sequences were defined in [5] via a polynomial extension of the Stern sequence. However, they satisfy recurrence relations that may be used as an alternative definition. To see this, let

$$\alpha_n = \frac{2^n - (-1)^n}{3}, \quad n \geq 0. \tag{1.1}$$

The polynomials $f_n(q)$ $(n \geq 0)$ and $\bar{f}_n(q)$ $(n \geq 2)$ satisfy (see [5]) the initial conditions

$$f_0(q) = 0, \quad f_1(q) = f_2(q) = \bar{f}_2(q) = 1 \tag{1.2}$$

and the recurrence

$$f_{n+1}(q) = q^{\alpha_{n-1}} f_n(q) + f_{n-1}(q), \ n \geq 1, \tag{1.3}$$

$$\bar{f}_{n+1}(q) = f_n(q) + q^{\alpha_n} f_{n-1}(q), \ n \geq 1. \tag{1.4}$$

Clearly, $f_n(1) = \bar{f}_n(1) = F_n$, where F_n in the nth Fibonacci number. Dilcher and Stolarsky showed [5] that both sequences f_n and $\bar{f}_n$ possess many properties analogous to the properties of the usual Fibonacci numbers. For instance, they proved the following identities for $k \geq 1$:

$$f_{2k+1}(q)f_{2k-1}(q^2) - qf_{2k}(q)f_{2k}(q^2) = 1, \tag{1.5}$$

$$f_{2k+1}(q)f_{2k+1}(q^2) - qf_{2k+2}(q)f_{2k}(q^2) = 1, \tag{1.6}$$

which generalize the well-known formula

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n.$$

The aim of this paper is to extend (1.5)–(1.6) and derive more quadratic relations for f_n and $\bar{f}_n$. In particular, we prove the following result.

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Theorem 1.1. *For any $r \geq 1$, $d \geq 1$, and $k \geq 0$, we have*

$$f_{2k+r}(q)f_{2k+2d-1}(q^{2^r}) - q^{\alpha_r} f_{2k+2d+r-1}(q)f_{2k}(q^{2^r}) = f_r(q)f_{2d-1}(q^{2^{2k+r}}), \tag{1.7}$$

$$f_{2k+2d+r}(q)f_{2k+1}(q^{2^r}) - q^{\alpha_r} f_{2k+r+1}(q)f_{2k+2d}(q^{2^r}) = f_r(q)f_{2d-1}(q^{2^{2k+r+1}}), \tag{1.8}$$

$$f_{2k+r}(q)f_{2k+2d-2}(q^{2^r}) - f_{2k+2d+r-2}(q)f_{2k}(q^{2^r}) = q^{2^r\alpha_{2k}} f_r(q)f_{2d-2}(q^{2^{2k+r}}), \tag{1.9}$$

$$f_{2k+2d+r-1}(q)f_{2k+1}(q^{2^r}) - f_{2k+r+1}(q)f_{2k+2d-1}(q^{2^r}) = q^{\alpha_{2k+r+1}} f_r(q)f_{2d-2}(q^{2^{2k+r+1}}), \tag{1.10}$$

$$f_{2k+r}(q)\bar{f}_{2k+2d}(q^{2^r}) - q^{\alpha_r} \bar{f}_{2k+2d+r}(q)f_{2k}(q^{2^r}) = f_r(q)\bar{f}_{2d}(q^{2^{2k+r}}), \tag{1.11}$$

$$\bar{f}_{2k+2d+r+1}(q)f_{2k+1}(q^{2^r}) - q^{\alpha_r} f_{2k+r+1}(q)\bar{f}_{2k+2d+1}(q^{2^r}) = f_r(q)\bar{f}_{2d}(q^{2^{2k+r+1}}), \tag{1.12}$$

$$f_{2k+r}(q)\bar{f}_{2k+2d+1}(q^{2^r}) - \bar{f}_{2k+2d+r+1}(q)f_{2k}(q^{2^r}) = q^{2^r\alpha_{2k}} f_r(q)\bar{f}_{2d+1}(q^{2^{2k+r}}), \tag{1.13}$$

$$\bar{f}_{2k+2d+r+2}(q)f_{2k+1}(q^{2^r}) - f_{2k+r+1}(q)\bar{f}_{2k+2d+2}(q^{2^r}) = q^{\alpha_{2k+r+1}} f_r(q)\bar{f}_{2d+1}(q^{2^{2k+r+1}}), \tag{1.14}$$

For $d = 1$, relations (1.7), (1.8), (1.11), and (1.12) take the simplest form, where the right-hand side depends only on $f_r(q)$. For instance, substituting $d = 1$ into (1.7) and (1.8) we obtain the following special case of Theorem 1.1: for any $r \geq 1$ and any $k \geq 0$,

$$f_{2k+r}(q)f_{2k+1}(q^{2^r}) - q^{\alpha_r} f_{2k+r+1}(q)f_{2k}(q^{2^r}) = f_r(q), \tag{1.15}$$

$$f_{2k+r+2}(q)f_{2k+1}(q^{2^r}) - q^{\alpha_r} f_{2k+r+1}(q)f_{2k+2}(q^{2^r}) = f_r(q). \tag{1.16}$$

Identities listed in Theorem 1.1 may be viewed as polynomial analogues of the following well-known formula for Fibonacci numbers:

$$F_{a+b}F_{a+c} - F_{a+b+c}F_a = (-1)^a F_b F_c$$

for all non-negative a, b and c.

We mention another consequence of the above identities. Relation (1.3) with n replaced by $n+1$ and relation (1.4) show that the polynomials f_n, $\bar{f}_{n+1}$ and f_{n+2} agree on terms up to (but not including) q^{α_n}. Thus, the following limits are well defined:

$$\begin{aligned} F(q) &= \lim_{n\to\infty} f_{2n}(q) = \lim_{n\to\infty} \bar{f}_{2n+1}(q) \\ &= 1 + q + q^2 + q^5 + q^6 + q^8 + q^9 + q^{10} + q^{21} + q^{22} + q^{24} + \cdots, \\ G(q) &= \lim_{n\to\infty} f_{2n+1}(q) = \lim_{n\to\infty} \bar{f}_{2n}(q) \\ &= 1 + q + q^3 + q^4 + q^5 + q^{11} + q^{12} + q^{13} + q^{16} + q^{17} + q^{19} + \cdots. \end{aligned}$$

Some properties of these formal power series were studied in [5]. Let us write (1.15) for odd and even r separately:

$$f_{2k+2m-1}(q)f_{2k+1}(q^{2^{2m-1}}) - q^{\alpha_{2m-1}} f_{2k+2m}(q)f_{2k}(q^{2^{2m-1}}) = f_{2m-1}(q), \tag{1.17}$$

$$f_{2k+2m}(q)f_{2k+1}(q^{2^{2m}}) - q^{\alpha_{2m}} f_{2k+2m+1}(q)f_{2k}(q^{2^{2m}}) = f_{2m}(q). \tag{1.18}$$

We fix m and let k go to infinity. Passing to a limit in (1.17) and (1.18) we obtain the following corollary.

Corollary 1.2. *For any $r \geq 1$ we have*

$$G(q)G(q^{2^{2m-1}}) - q^{\alpha_{2m-1}} F(q)F(q^{2^{2m-1}}) = f_{2m-1}(q), \tag{1.19}$$

$$F(q)G(q^{2^{2m}}) - q^{\alpha_{2m}} G(q)F(q^{2^{2m}}) = f_{2m}(q). \tag{1.20}$$

The same corollary can be also deduced from any of the identities (1.7), (1.8), (1.11) or (1.12). Identity (1.20) was obtained first in [5, Proposition 5.3] by a different method. Relation (1.19) for $r \geq 2$ seems to be new.

In addition to Theorem 1.1, there are other identities satisfied by f_n and $\bar{f}_n$. Dilcher and Stolarsky [5, Proposition 4.2] also found the following ones:

$$f_{2k+1}(q)\bar{f}_{2k-1}(q^2) - \bar{f}_{2k}(q)f_{2k}(q^2) = q^{\beta_{2k}-1}, \quad k \geq 2, \tag{1.21}$$

$$\bar{f}_{2k+2}(q)f_{2k}(q^2) - f_{2k+1}(q)\bar{f}_{2k+1}(q^2) = -q^{\alpha_{2k+1}-1}, \quad k \geq 1, \tag{1.22}$$

where the numbers α_n are defined by (1.1) and

$$\beta_n = \frac{5 \cdot 2^{n-2} + (-1)^n}{3}, \quad n \geq 2.$$

Both (1.21) and (1.22) fit into a general family of identities. However, in contrast to Theorem 1.1 the equations become more complicated. We prove the following result.

Theorem 1.3. *For $r \geq 1$, $d \geq 1$, and $k \geq 1$, we have*

$$\begin{aligned} &f_{2k+2d+r-2}(q)\bar{f}_{2k+1}(q^{2^r}) - \bar{f}_{2k+r+1}(q)f_{2k+2d-2}(q^{2^r}) \\ &\quad = q^{2^r\alpha_{2k}} f_r(q) \cdot \left(f_{2d-1}(q^{2^{2k+r-1}}) - f_{2d-2}(q^{2^{2k+r}})\right). \end{aligned} \tag{1.23}$$

Indeed, Theorem 1.3 extends (1.21) and (1.22). To see this, notice that $2\alpha_{2k} = \alpha_{2k+1} - 1$, $f_1(q) = f_1(q^{2^{2k}}) = 1$ and $f_0(q^{2^{2k+1}}) = 0$. Hence, multiplying both sides of (1.23) by -1 and setting $r = 1$, $d = 1$ we deduce (1.22).

Now take $k \geq 2$ and write (1.23) for $k-1$ instead of k. Substituting $r = 1$ and $d = 2$ we obtain

$$\begin{aligned} f_{2k+1}(q)\bar{f}_{2k-1}(q^2) - \bar{f}_{2k}(q)f_{2k}(q^2) &= q^{2\alpha_{2k-2}} f_1(q) \left(f_3(q^{2^{2k-2}}) - f_2(q^{2^{2k-1}})\right) \\ = q^{2\alpha_{2k-2}}(1 + q^{2^{2k-2}} - 1) &= q^{2\alpha_{2k-2}+2^{2k-2}} = q^{\beta_{2k}-1}, \end{aligned}$$

which gives (1.21).

Further relations of similar type are also discussed in Section 3.

2. Proofs of the Identities

Before proving Theorem 1.1 we state an easy property of the sequence α_n.

Lemma 2.1. *For any $s \geq 0$ and $m \geq 0$, we have*

$$\alpha_{m+s} = 2^s\alpha_m + (-1)^m\alpha_s. \tag{2.1}$$

Proof of Theorem 1.1. (i) First, we prove (1.7)–(1.10). The proof is by induction on d.

Let $d = 1$. The identities (1.9)–(1.10) become trivial since both their sides vanish identically. Notice that for $d = 1$ equations (1.7)–(1.8) become (1.15)–(1.16), respectively. We prove the two latter identities by induction on k.

Relation (1.15) for $k = 0$ follows trivially from (1.2), while (1.16) for $k = 0$ becomes $f_{r+2}(q) - q^{\alpha_r} f_{r+1}(q) = f_r(q)$, which is a consequence of (1.3). Suppose now that both identities (1.15) and (1.16) hold up to $k-1$. Using (1.3) and (2.1) with $m = 2k-1$, $s = r$, we rewrite the left hand side of (1.15) as

$$\begin{aligned} &f_{2k+r}(q)\left(q^{2^r\alpha_{2k-1}} f_{2k}(q^{2^r}) + f_{2k-1}(q^{2^r})\right) \\ &\qquad - q^{\alpha_r}\left(q^{\alpha_{2k+r-1}} f_{2k+r}(q) + f_{2k+r-1}(q)\right) f_{2k}(q^{2^r}) \\ &\quad = f_{2k+r}(q)f_{2k-1}(q^{2^r}) - q^{\alpha_r} f_{2k+r-1}(q)f_{2k}(q^{2^r}) = f_r(q). \end{aligned}$$

The last equality follows from (1.16) for $k-1$ in place of k. In a similar way, using (1.3) and (2.1) with $m = 2k$, $s = r$, we can rewrite the left hand side of (1.16) as

$$\begin{aligned}&\left(q^{\alpha_{2k+r}} f_{2k+r+1}(q) + f_{2k+r}(q)\right) f_{2k+1}(q^{2^r})\\ &\qquad -q^{\alpha_r} f_{2k+r+1}(q)\left(q^{2^r\alpha_{2k}} f_{2k+1}(q^{2^r}) + f_{2k}(q^{2^r})\right)\\ &\quad = f_{2k+r}(q)f_{2k+1}(q^{2^r}) - q^{\alpha_r} f_{2k+r+1}(q) f_{2k}(q^{2^r}) = f_r(q)\end{aligned}$$

by (1.15) for k, which has already been obtained above. Thus we proved (1.15)–(1.16). Hence, (1.7)–(1.10) hold for $d = 1$.

Now let $d \geq 2$ and assume that we proved (1.7)–(1.10) up to $d-1$.

We start with (1.9) and (1.10). Using (1.3) and then (2.1) for $m = 2k+2d-4$, $s = r$, we rewrite the left-hand side of (1.9):

$$\begin{aligned}&f_{2k+r}(q)\left(q^{2^r\alpha_{2k+2d-4}} f_{2k+2d-3}(q^{2^r}) + f_{2k+2d-4}(q^{2^r})\right)\\ &\quad - \left(q^{\alpha_{2k+2d+r-4}} f_{2k+2d+r-3}(q) + f_{2k+2d+r-4}(q)\right) f_{2k}(q^{2^r})\\ &= q^{2^r\alpha_{2k+2d-4}}\left(f_{2k+r}(q)f_{2k+2d-3}(q^{2^r}) - q^{\alpha_r} f_{2k+2d+r-3}(q) f_{2k}(q^{2^r})\right)\\ &\quad + f_{2k+r}(q)f_{2k+2d-4}(q^{2^r}) - f_{2k+2d+r-4}(q)f_{2k}(q^{2^r}).\end{aligned}$$

Using the induction hypothesis for (1.7) and (1.9) and then applying (2.1) for $m = 2d-4$, $s = 2k$, we continue as follows:

$$\begin{aligned}&= q^{2^r\alpha_{2k+2d-4}} f_r(q) f_{2d-3}(q^{2^{2k+r}}) + q^{2^r\alpha_{2k}} f_r(q) f_{2d-4}(q^{2^{2k+r}})\\ &= q^{2^r\alpha_{2k}} f(r)\left(\left(q^{2^{2k+r}}\right)^{\alpha_{2d-4}} f_{2d-3}(q^{2^{2k+r}}) + f_{2d-4}(q^{2^{2k+r}})\right)\\ &= q^{2^r\alpha_{2k}} f(r) f_{2d-2}(q^{2^{2k+r}})\end{aligned}$$

by (1.3). This proves (1.9) for d.

In a similar way, using (1.3) and then (2.1) with $m = 2k+2d-3$, $s = r$, we rewrite the left-hand side of (1.10):

$$\begin{aligned}&\left(q^{\alpha_{2k+2d+r-3}} f_{2k+2d+r-2}(q) + f_{2k+2d+r-3}(q)\right) f_{2k+1}(q^{2^r})\\ &\quad -f_{2k+r+1}(q)\left(q^{2^r\alpha_{2k+2d-3}} f_{2k+2d-2}(q^{2^r}) + f_{2k+2d-3}(q^{2^r})\right)\\ &= q^{\alpha_{2k+2d+r-3}}\left(f_{2k+2(d-1)+r}(q) f_{2k+1}(q^{2^r}) - q^{\alpha_r} f_{2k+r+1}(q) f_{2k+2(d-1)}(q^{2^r})\right)\\ &\quad + f_{2k+2(d-1)+r-1}(q) f_{2k+1}(q^{2^r}) - f_{2k+r+1}(q) f_{2k+2(d-1)-1}(q^{2^r}).\end{aligned}$$

Using the induction hypothesis for (1.8) and (1.10) and applying (2.1) with $m = 2d-4$, $s = 2k+r+1$, we continue as follows:

$$\begin{aligned}&= q^{\alpha_{2k+2d+r-3}} f_r(q) f_{2d-3}(q^{2^{2k+r+1}}) + q^{\alpha_{2k+r+1}} f_r(q) f_{2d-4}(q^{2^{2k+r+1}})\\ &= q^{\alpha_{2k+r+1}} f(r)\left(\left(q^{2^{2k+r+1}}\right)^{\alpha_{2d-4}} f_{2d-3}(q^{2^{2k+r+1}}) + f_{2d-4}(q^{2^{2k+r+1}})\right)\\ &= q^{\alpha_{2k+r+1}} f(r) f_{2d-2}(q^{2^{2k+r+1}})\end{aligned}$$

by (1.3). This proves (1.10) for d.

Now we prove (1.7). Using (1.3) and then (2.1) for $m = 2k + 2d - 3$, $s = r$, we rewrite the left-hand side of (1.7):

$$\begin{aligned}
&f_{2k+r}(q)\left(q^{2^r\alpha_{2k+2d-3}} f_{2k+2d-2}(q^{2^r}) + f_{2k+2d-3}(q^{2^r})\right)\\
&\quad -q^{\alpha_r}\left(q^{\alpha_{2k+2d+r-3}} f_{2k+2d+r-2}(q) + f_{2k+2d+r-3}(q)\right) f_{2k}(q^{2^r})\\
&= q^{2^r\alpha_{2k+2d-3}}\left(f_{2k+r}(q) f_{2k+2d-2}(q^{2^r}) - f_{2k+2d+r-2}(q) f_{2k}(q^{2^r})\right)\\
&\quad + f_{2k+r}(q) f_{2k+2d-3}(q^{2^r}) - q^{\alpha_r} f_{2k+2d+r-3}(q) f_{2k}(q^{2^r}).
\end{aligned}$$

Recall that we have already established (1.9) for d. Using it together with the induction hypothesis for (1.7) and with (2.1) for $m = 2d-3$, $s = 2k$, we continue the above transformation:

$$\begin{aligned}
&= q^{2^r\alpha_{2k+2d-3}} \cdot q^{2^r\alpha_{2k}} f_r(q) f_{2d-2}(q^{2^{2k+r}}) + f_r(q) f_{2d-3}(q^{2^{2k+r}})\\
&= f(r)\left(\left(q^{2^{2k+r}}\right)^{\alpha_{2d-3}} f_{2d-2}(q^{2^{2k+r}}) + f_{2d-3}(q^{2^{2k+r}})\right) = f(r) f_{2d-1}(q^{2^{2k+r}})
\end{aligned}$$

by (1.3). This proves (1.7) for d.

Finally, we prove (1.8). Using (1.3) and then (2.1) for $m = 2k + 2d - 2$, $s = r$, we rewrite the left-hand side of (1.8):

$$\begin{aligned}
&\left(q^{\alpha_{2k+2d+r-2}} f_{2k+2d+r-1}(q) + f_{2k+2d+r-2}(q)\right) f_{2k+1}(q^{2^r})\\
&\quad -q^{\alpha_r} f_{2k+r+1}(q)\left(q^{2^r\alpha_{2k+2d-2}} f_{2k+2d-1}(q^{2^r}) + f_{2k+2d-2}(q^{2^r})\right)\\
&= q^{\alpha_{2k+2d+r-2}}\left(f_{2k+2d+r-1}(q) f_{2k+1}(q^{2^r}) - f_{2k+r+1}(q) f_{2k+2d-1}(q^{2^r})\right)\\
&\quad + f_{2k+2(d-1)+r}(q) f_{2k+1}(q^{2^r}) - q^{\alpha_r} f_{2k+r+1}(q) f_{2k+2(d-1)}(q^{2^r}).
\end{aligned}$$

Recall that we have already established (1.10) for d. Using it together with the induction hypothesis for (1.8) and with (2.1) for $m = 2d - 3$, $s = 2k + r + 1$, we continue the above transformation:

$$\begin{aligned}
&= q^{\alpha_{2k+2d+r-2}} \cdot q^{\alpha_{2k+r+1}} f_r(q) f_{2d-2}(q^{2^{2k+r+1}}) + f_r(q) f_{2d-3}(q^{2^{2k+r+1}})\\
&= \left(\left(q^{2^{2k+r+1}}\right)^{\alpha_{2d-3}} f_{2d-2}(q^{2^{2k+r+1}}) + f_{2d-3}(q^{2^{2k+r+1}})\right)\\
&= f_{2d-1}(q^{2^{2k+r+1}})
\end{aligned}$$

by (1.3). This proves (1.8) for d. Thus, we completed the proof of (1.7)–(1.10).

(ii) To prove (1.11) we multiply both sides of (1.9) by $q^{2^r\alpha_{2k+2d-1}}$ and add to (1.7). Taking into account (2.1) with $m = 2k + 2d - 1$ and $s = r$ and (1.4), we transform the left-hand side of that sum:

$$\begin{aligned}
&f_{2k+r}(q)\left(f_{2k+2d-1}(q^{2^r}) + (q^{2^r})^{\alpha_{2k+2d-1}} f_{2k+2d-2}(q^{2^r})\right)\\
&\quad -q^{\alpha_r}\left(f_{2k+2d+r-1}(q) + q^{\alpha_{2k+2d+r-1}} f_{2k+2d+r-2}(q)\right) f_{2k}(q^{2^r})\\
&= f_{2k+r}(q)\bar{f}_{2k+2d}(q^{2^r}) - q^{\alpha_r}\bar{f}_{2k+2d+r}(q) f_{2k}(q^{2^r}).
\end{aligned}$$

The right-hand side of the above sum under consideration equals

$$\begin{aligned}
&f_r(q)\left(f_{2d-1}(q^{2^{2k+r}}) + q^{2^r\alpha_{2k+2d-1}} \cdot q^{2^r\alpha_{2k}} f_{2d-2}(q^{2^{2k+r}})\right)\\
&= f_r(q)\left(f_{2d-1}(q^{2^{2k+r}}) + \left(q^{2^{2k+r}}\right)^{\alpha_{2d-1}} f_{2d-2}(q^{2^{2k+r}})\right)\\
&= f_r(q)\bar{f}_{2d}(q^{2^{2k+r}})
\end{aligned}$$

by (2.1) with $m = 2d - 1$, $s = 2k$ and by (1.4). This proves (1.11).

The remaining three identities can be proved in a similar way. For instance, to prove (1.12) we add (1.10) multiplied by $q^{\alpha_{2k+2d+r}}$ to (1.8). To prove (1.13) we add (1.9) with d replaced by $d+1$ to (1.7) multiplied by $q^{2^r\alpha_{2k+2d}}$. Finally, to obtain (1.14) we add (1.10) with d replaced by $d+1$ to (1.8) multiplied by $q^{\alpha_{2k+2d+r+1}}$. We leave the details to the reader. □

Proof of Theorem 1.3. Let $k \geq 1$. First, substitute $k-1$ in place of k in (1.8), multiply both sides of the result by $q^{2^r\alpha_{2k}}$ and subtract (1.9). Taking into account (2.1) with $m = 2k$, $s = r$ and (1.4), we see that the left-hand side of the resulting relation equals

$$\begin{aligned} & f_{2k+2d+r-2}(q)\left(\left(q^{2^r}\right)^{\alpha_{2k}} f_{2k-1}(q^{2^r}) + f_{2k}(q^{2^r})\right) \\ &\quad - \left(q^{\alpha_{2k+r}} f_{2k+r-1}(q) + f_{2k+r}(q)\right) f_{2k+2d-2}(q^{2^r}) \\ &= f_{2k+2d+r-2}(q)\bar{f}_{2k+1}(q^{2^r}) - \bar{f}_{2k+r+1}(q) f_{2k+2d-2}(q^{2^r}), \end{aligned}$$

while the right-hand side is

$$q^{2^r\alpha_{2k}} f_r(q)\left(f_{2d-1}(q^{2^{2k+r-1}}) - f_{2d-2}(q^{2^{2k+r}})\right).$$

This proves Theorem 1.3. □

3. Further Identities and Concluding Remarks

In this section we list a few related identities. Their proofs are similar to those given above and the details are left to the reader. For instance, the following analogues of (1.23) can be deduced from (1.7)–(1.10). For any $d \geq 1$, $r \geq 1$ and $k \geq 0$, we have

$$\begin{aligned} & \bar{f}_{2k+r+2}(q) f_{2k+2d-1}(q^{2^r}) - f_{2k+2d+r-1}(q)\bar{f}_{2k+2}(q^{2^r}) \\ &\quad = q^{\alpha_{2k+r+1}} f_r(q)\left(f_{2d-1}(q^{2^{2k+r}}) - f_{2d-2}(q^{2^{2k+r+1}})\right), \\ & \bar{f}_{2k+r+3}(q) f_{2k+2d+1}(q^{2^r}) - q^{\alpha_r} f_{2k+2d+r+1}(q)\bar{f}_{2k+3}(q^{2^r}) \\ &\quad = f_r(q)\left(f_{2d-1}(q^{2^{2k+r+2}}) - q^{2^{2k+r+1}} f_{2d}(q^{2^{2k+r+1}})\right), \\ & f_{2k+2d+r}(q)\bar{f}_{2k+2}(q^{2^r}) - q^{\alpha_r}\bar{f}_{2k+r+2}(q) f_{2k+2d}(q^{2^r}) \\ &\quad = f_r(q)\left(f_{2d-1}(q^{2^{2k+r+1}}) - q^{2^{2k+r}} f_{2d}(q^{2^{2k+r}})\right). \end{aligned}$$

In a similar way, the following identities can be deduced from (1.11)–(1.14):

$$\begin{aligned} & \bar{f}_{2k+r+2}(q)\bar{f}_{2k+2d+2}(q^{2^r}) - \bar{f}_{2k+2d+r+2}(q)\bar{f}_{2k+2}(q^{2^r}) \\ &\quad = q^{\alpha_{2k+r+1}} f_r(q)\left(\bar{f}_{2d+2}(q^{2^{2k+r}}) - \bar{f}_{2d+1}(q^{2^{2k+r+1}})\right), \\ & \bar{f}_{2k+r+3}(q)\bar{f}_{2k+2d+2}(q^{2^r}) - q^{\alpha_r}\bar{f}_{2k+2d+r+2}(q)\bar{f}_{2k+3}(q^{2^r}) \\ &\quad = f_r(q)\left(\bar{f}_{2d}(q^{2^{2k+r+2}}) - q^{2^{2k+r+1}}\bar{f}_{2d+1}(q^{2^{2k+r+1}})\right), \\ & \bar{f}_{2k+2d+r+1}(q)\bar{f}_{2k+2}(q^{2^r}) - q^{\alpha_r}\bar{f}_{2k+r+2}(q)\bar{f}_{2k+2d+1}(q^{2^r}) \\ &\quad = f_r(q)\left(\bar{f}_{2d}(q^{2^{2k+r+1}}) - q^{2^{2k+r}}\bar{f}_{2d+1}(q^{2^{2k+r}})\right), \\ & \bar{f}_{2k+2d+r+3}(q)\bar{f}_{2k+3}(q^{2^r}) - \bar{f}_{2k+r+3}(q)\bar{f}_{2k+2d+3}(q^{2^r}) \\ &\quad = q^{2^r\alpha_{2k+2}} f_r(q)\left(\bar{f}_{2d+2}(q^{2^{2k+r+1}}) - \bar{f}_{2d+1}(q^{2^{2k+r+2}})\right). \end{aligned}$$

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