WHAT FIBONACCI NUMBERS HAVE TO DO WITH CONGRUENT NUMBERS?

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Abstract. There is no right triangle with rational sides and area equal to one. We study a convergent sequence of "Fibonacci ratios" and show that its limit is related to rational right triangles of area as close to one as one may like.

1. Introduction: One is Not a Congruent Number

A more than one-thousand-year-old Arab manuscript deals with the following problem: given an integer n, find a square x^2 such that $x^2 \pm n$ are both squares (cf. [1]). It is easy to see that the appearance of three squares in arithmetic progression of common difference n is equivalent to the existence of a right triangle with rational sides and area equal to n. A positive integer n is said to be a *congruent number* if there exists a right triangle with rational sides and area equal to n. Fibonacci found the following right triangle with sides

$$\frac{3}{2}, \ \frac{20}{3}, \ \frac{41}{6}, \tag{1.1}$$

which implies that 5 is a congruent number. It is easy to construct congruent numbers from pythagorean triples; for example, the triple 3–4–5 leads to the congruent number 6. However, it seems rather difficult to check whether a given positive integer is a congruent number or not (since the sides of the right triangle are not necessarily integers). Using his famous *descent infinie*, Fermat proved that 1, 2, and 3 are not congruent numbers. In particular, there is no perfect squares amongst the congruent numbers (since otherwise the corresponding rational triangle would be similar to one with area equal to 1).

The congruent number problem is to decide whether a given positive integer is a congruent number. There is a simple translation of the congruent number problem into the theory of elliptic curves (see [2]) which implies that n is a congruent number if and only if the elliptic curve E_n given by the Weierstrass equation $Y^2 = X^3 - n^2X$ contains a rational point (x, y) with non-vanishing y-coordinate or, equivalently, that the Mordell-Weil group $E_n(\mathbb{Q})$ of rational points has positive rank. Celebrated work of Tunnell [4] implies that if n is an odd square-free congruent number, then the number of integer representations of n by the quadratic form $2x^2 + y^2 + 8z^2$ is twice the number of integer representations of n by $2x^2 + y^2 + 32z^2$, and the converse implication is true provided the famous, yet unsolved Birch & Swinnerton-Dyer conjecture holds (i.e., the rank of E_n is positive if and only if the associated L-function vanishes at the central point). A similar criterion exists for even numbers n. This conjectural equivalent for the congruent number problem in terms of the number of representations of n by certain ternary quadratic forms allows us to determine by computation whether a given integer n is congruent or not.

Fibonacci sequences are sequences built from Fibonacci numbers; such sequences have been studied for various reasons. In this paper, we shall study a new Fibonacci sequence which has a bearing on congruent numbers.

2. A Sequence of Fibonacci Ratios

In view of Fibonacci's example (1.1) we first construct a sequence of rational numbers which converges to $\sqrt{5}$; the elements of this sequence are explicitly given in terms of Fibonacci numbers F_n which are recursively defined by $F_{n+1} = F_n + F_{n-1}$ and $F_1 = 1, F_0 = 0$. We start with the continued fraction expansion of the quadratic irrational

$$\sqrt{5} = 2 + \cfrac{1}{4 + \frac{1}{4+\cdots}} = [2, \overline{4}],$$

where we use the standard notation for continued fractions and $\overline{4}$ indicates that all partial denominators in the infinite expansion of the fractional part of $\sqrt{5}$ are equal to 4. For the sake of completeness we mention that, if we let $x = [2, \overline{4}]$, then the continued fraction expansion of $\sqrt{5}$ follows from solving the quadratic equation

$$x - 2 = \frac{1}{x+2} \qquad \Longleftrightarrow \qquad x^2 = 5;$$

obviously, the positive root of the quadratic equation yields the value for x.

Theorem 2.1. *The nth convergent of the continuous fraction expansion of $\sqrt{5}$ is given by*

$$r_n := \frac{F_{3n+1} + F_{3n-1}}{F_{3n}} \quad \textit{for} \quad n \in \mathbb{N}.$$

Moreover,

$$r_n = \sqrt{5} + O(\phi^{-6n}),$$

where $\phi := \frac{1}{2}(\sqrt{5} + 1)$ is the golden ratio.

Proof. It is well-known that the convergents

$$\frac{p_n}{q_n} = [2, 4, \ldots, 4] \quad \text{(with } n \text{ partial quotients 4)}$$

yield successively the closest rational approximations to $\sqrt{5}$ with respect to the size of their denominators. Actually, this holds for arbitrary real numbers in place of $\sqrt{5}$ and is known as Lagrange's law of best approximations; for this and further results on continued fractions we refer to [3]. Moreover, the sequences of numerators p_n and denominators q_n, respectively, satisfy the following linear recurrence: for $n \in \mathbb{N}$,

$$\begin{aligned} p_{n+1} &= 4p_n + p_{n-1} \qquad \text{with} \qquad p_0 = 2,\ p_{-1} = 1, \\ q_{n+1} &= 4q_n + q_{n-1} \qquad \text{with} \qquad q_0 = 1,\ q_{-1} = 0. \end{aligned}$$

This yields a quickly converging sequence of rational numbers: $\frac{2}{1}, \frac{9}{4}, \frac{38}{17}, \ldots \to \sqrt{5}$. In order to solve the linear recurrence we first factorize the characteristic polynomial

$$X^2 - 4X - 1 = (X - \alpha)(X - \beta) \quad \text{with} \quad \alpha = 2 + \sqrt{5},\ \beta = 2 - \sqrt{5}.$$

Any solution of the recurrence $x_{n+1} = 4x_n + x_{n-1}$ is a linear combination of α^n and β^n. Writing $p_n = a\alpha^n + b\beta^n$, we deduce in view of the starting values the linear equations

$$a + b = 1 \quad \text{and} \quad a\alpha + b\beta = 2,$$

which yields the coefficients $a = b = \frac{1}{2}$. Thus, we obtain the explicit representation

$$p_n = \frac{1}{2}((2+\sqrt{5})^n + (2-\sqrt{5})^n) = \frac{1}{2}F_{3n} + F_{3n-1};$$

the latter identity follows easily by induction from Binet's formula for the Fibonacci numbers,

$$F_n = \frac{1}{\sqrt{5}}(\phi^n - (-\phi)^{-n}). \tag{2.1}$$

By the same reasoning for the sequence of denominators q_n we find

$$q_n = \frac{1}{2\sqrt{5}}((2+\sqrt{5})^n - (2-\sqrt{5})^n) = \frac{1}{2}F_{3n}.$$

Hence,

$$r_n = \frac{p_n}{q_n} = 1 + 2\frac{F_{3n-1}}{F_{3n}} = \frac{F_{3n+1}+F_{3n-1}}{F_{3n}}.$$

Alternatively, one can obtain a representation in terms of values of Chebyshev poylnomials. (For this and the above expression in terms of Fibonacci numbers we refer to Sloane's *On-Line Encyclopedia of Integer Sequences* at `www.research.att.com/njas/sequences/`.) It remains to estimate the speed of convergence. Taking Binet's formula (2.1) into account, a short computation leads to

$$r_n = 1 + 2\frac{\phi^{3n-1} + (-1)^{3n}\phi^{-3n+1}}{\phi^{3n} + (-1)^{3n-1}\phi^{-3n}} = 1 + 2\phi^{-1} + O(\phi^{-6n}) = \sqrt{5} + O(\phi^{-6n}).$$

□

Theorem 2.2. *For any $n \in \mathbb{N}$, let*

$$a_n = \frac{3}{2r_n}, \quad b_n = \frac{20}{3r_n}, \quad \text{and } c_n = \frac{41}{6r_n},$$

where the r_n are non-zero rational numbers defined in Theorem 2.1. Then the numbers a_n, b_n, c_n define a rational right triangle of area $1 + O(\phi^{-6n})$.

Proof. Since a_n, b_n, c_n are a non-zero rational multiple of Fibonacci's triple (1.1), these numbers yield a rational right triangle. In view of Theorem 2.1 its area is easily computed as

$$\frac{1}{2}a_nb_n = \frac{5}{r_n^2} = \frac{5}{(\sqrt{5}+O(\phi^{-6n}))^2} = 1 + O(\phi^{-6n}).$$

□

Remark. *Consider the amplitude of light which is a physical bound for the size of objects which human eyes can see, or the Planck length $l_P \approx 1.616\ldots \times 10^{-35}$ meter which is the smallest size of objects in quantum mechanics. We can construct a rational triangle having an area which differs from one by a quantity that the naked eye cannot see. To have an approximation error less than the Planck constant we would have to choose only $n \geq 28$.*

We cannot resist to illustrate the above construction with examples. For $n = 10$ we obtain the sides $a_{10} = \frac{208\,010}{310\,083}, b_{10} = \frac{8\,320\,400}{2\,790\,747}, c_{10} = \frac{208\,010}{68\,067}$, giving the area

$$\frac{5}{r_{10}^2} = 5\left(\frac{F_{30}}{F_{31}+F_{29}}\right)^2 = \frac{865\,363\,202\,000}{865\,363\,202\,001} = 0.99999\,99999\,98844\,41585\ \ldots.$$

For $n = 11$ we get the sides $a_{11} = \frac{5\,286\,867}{7\,881\,196}, b_{11} = \frac{17\,622\,890}{5\,910\,897}, c_{11} = \frac{72\,253\,849}{23\,643\,588}$, giving the area

$$\frac{5}{r_{11}^2} = 5\left(\frac{F_{33}}{F_{34}+F_{32}}\right)^2 = \frac{15\,528\,312\,597\,605}{15\,528\,312\,597\,604} = 1.\,00000\,00000\,00064\,39850\ \ldots .$$

Of course, we may argue with $\sqrt{n^2+1} = [n, \overline{2n}]$ and $\sqrt{n^2+2} = [n, \overline{n, 2n}]$ or other continued fraction expansions for irrationalities in order to obtain rational right triangles with areas as close to an arbitrarily given positive integer as we want, however, we leave this task to the interested reader.

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References

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