CONVOLUTIONS OF TRIBONACCI, FUSS–CATALAN, AND MOTZKIN SEQUENCES

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Abstract. We introduce a class of sequences, defined by means of partial Bell polynomials, that contains a basis for the space of linear recurrence sequences with constant coefficients as well as other well-known sequences like Catalan and Motzkin. For the family of 'Bell sequences' considered in this paper, we give a general multifold convolution formula and illustrate our result with a few explicit examples.

1. Introduction

Given numbers a and b, not both equal to zero, and given a sequence $c_1, c_2, \dots$, we consider the sequence (y_n) given by

$$y_0 = 1, \quad y_n = \sum_{k=1}^{n} \binom{an+bk}{k-1} \frac{(k-1)!}{n!} B_{n,k}(1!c_1, 2!c_2, \dots) \text{ for } n \geq 1, \tag{1.1}$$

where $B_{n,k}$ denotes the (n,k)-th partial Bell polynomial defined as

$$B_{n,k}(x_1, \dots, x_{n-k+1}) = \sum_{\alpha \in \pi(n,k)} \frac{n!}{\alpha_1! \alpha_2! \cdots \alpha_{n-k+1}!} \left(\tfrac{x_1}{1!}\right)^{\alpha_1} \cdots \left(\tfrac{x_{n-k+1}}{(n-k+1)!}\right)^{\alpha_{n-k+1}}$$

with $\pi(n,k)$ denoting the set of multi-indices $\alpha \in \mathbb{N}_0^{n-k+1}$ such that $\alpha_1 + \alpha_2 + \cdots = k$ and $\alpha_1 + 2\alpha_2 + 3\alpha_3 + \cdots = n$. For more about Bell polynomials, see e.g. [4, Chapter 3]. In general, there is no need to impose any restriction on the entries $x_1, x_2, \dots$ other than being contained in a commutative ring. Here we are mainly interested in $\mathbb{Z}$ and $\mathbb{Z}[x]$.

The class of sequences (1.1) turns out to offer a unified structure to a wide collection of known sequences. For instance, with $a = 0$ and $b = 1$, any linear recurrence sequence with constant coefficients $c_1, c_2, \dots, c_d$, can be written as a linear combination of sequences of the form (1.1). In fact, if (a_n) is a recurrence sequence satisfying $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_d a_{n-d}$ for $n \geq d$, then there are constants $\lambda_0, \lambda_1, \dots, \lambda_{d-1}$ (depending on the initial values of the sequence) such that $a_n = \lambda_0 y_n + \lambda_1 y_{n-1} + \cdots + \lambda_{d-1} y_{n-d+1}$ with

$$y_0 = 1, \quad y_n = \sum_{k=1}^{n} \frac{k!}{n!} B_{n,k}(1!c_1, 2!c_2, \dots) \text{ for } n \geq 1.$$

For more details about this way of representing linear recurrence sequences, cf. [3].

On the other hand, if $a = 1$ and $b = 0$, we obtain sequences like Catalan and Motzkin by making appropriate choices of c_1 and c_2, and by setting $c_j = 0$ for $j \geq 3$. These and other concrete examples will be discussed in sections 3 and 4.

In this paper, we focus on convolutions and will use known properties of the partial Bell polynomials to prove a multifold convolution formula for (1.1).

2. Convolution Formula

Our main result is the following formula.

Theorem 2.1. *Let $y_0 = 1$ and for $n \ge 1$,*

$$y_n = \sum_{k=1}^{n} \binom{an+bk}{k-1} \frac{(k-1)!}{n!} B_{n,k}(1!c_1, 2!c_2, \dots).$$

For $r \ge 1$, we have

$$\sum_{m_1+\cdots+m_r=n} y_{m_1} \cdots y_{m_r} = r \sum_{k=1}^{n} \binom{an+bk+r-1}{k-1} \frac{(k-1)!}{n!} B_{n,k}(1!c_1, 2!c_2, \dots). \tag{2.1}$$

In order to prove this theorem, we recall a convolution formula for partial Bell polynomials that was given by the authors in [2, Section 3, Corollary 11].

Lemma 2.2. *Let $\alpha(\ell, m)$ be a linear polynomial in ℓ and m. For any $\tau \neq 0$, we have*

$$\sum_{\ell=0}^{k} \sum_{m=\ell}^{n} \frac{\binom{\alpha(\ell,m)}{k-\ell}\binom{\tau-\alpha(\ell,m)}{\ell}\binom{n}{m}}{\alpha(\ell,m)(\tau-\alpha(\ell,m))\binom{k}{\ell}} B_{m,\ell} B_{n-m,k-\ell} = \frac{\tau - \alpha(0,0) + \alpha(k,n)}{\tau\alpha(k,n)(\tau-\alpha(0,0))} \binom{\tau}{k} B_{n,k}.$$

This formula is key for proving Theorem 2.1. For illustration purposes, we start by proving the special case of a simple convolution (i.e. $r = 2$).

Lemma 2.3. *The sequence (y_n) defined by (1.1) satisfies*

$$\sum_{m=0}^{n} y_m\, y_{n-m} = 2 \sum_{k=1}^{n} \binom{an+bk+1}{k-1} \frac{(k-1)!}{n!} B_{n,k}(1!c_1, 2!c_2, \dots).$$

Proof. We begin by assuming $a, b \ge 0$. For $n \ge 0$ we can rewrite y_n as

$$y_n = \sum_{k=0}^{n} \frac{1}{an+bk+1} \binom{an+bk+1}{k} \frac{k!}{n!} B_{n,k}(1!c_1, 2!c_2, \dots). \tag{2.2}$$

By definition,

$$\begin{aligned}
&\sum_{m=0}^{n} y_m\, y_{n-m} \\
&\quad = \sum_{m=0}^{n} \left[\sum_{\ell=0}^{m} \tfrac{1}{am+b\ell+1} \tbinom{am+b\ell+1}{\ell} \tfrac{\ell!}{m!} B_{m,\ell} \right] \left[\sum_{j=0}^{n-m} \tfrac{1}{a(n-m)+bj+1} \tbinom{a(n-m)+bj+1}{j} \tfrac{j!}{(n-m)!} B_{n-m,j} \right] \\
&\quad = \sum_{m=0}^{n} \sum_{k=0}^{n} \sum_{\ell=0}^{k} \frac{\binom{am+b\ell+1}{\ell}\binom{a(n-m)+b(k-\ell)+1}{k-\ell}}{(am+b\ell+1)(a(n-m)+b(k-\ell)+1)} \frac{\ell!}{m!} \frac{(k-\ell)!}{(n-m)!} B_{m,\ell} B_{n-m,k-\ell} \\
&\quad = \sum_{k=0}^{n} \frac{k!}{n!} \sum_{\ell=0}^{k} \sum_{m=\ell}^{n} \frac{\binom{a(n-m)+b(k-\ell)+1}{k-\ell}\binom{am+b\ell+1}{\ell}\binom{n}{m}}{(am+b\ell+1)(a(n-m)+b(k-\ell)+1)\binom{k}{\ell}} B_{m,\ell} B_{n-m,k-\ell} \\
&\quad = \sum_{k=0}^{n} \frac{k!}{n!} \left[\sum_{\ell=0}^{k} \sum_{m=\ell}^{n} \frac{\binom{\alpha(\ell,m)}{k-\ell}\binom{\tau-\alpha(\ell,m)}{\ell}\binom{n}{m}}{(\tau-\alpha(\ell,m))\alpha(\ell,m)\binom{k}{\ell}} B_{m,\ell} B_{n-m,k-\ell} \right]
\end{aligned}$$

with $\alpha(\ell, m) = a(n-m) + b(k-\ell) + 1$ and $\tau = an + bk + 2$.

Thus, by Lemma 2.2,

$$\begin{aligned}\sum_{m=0}^{n} y_m\, y_{n-m} &= \sum_{k=0}^{n} \frac{k!}{n!}\left[\frac{\tau-\alpha(0,0)+\alpha(k,n)}{\tau\alpha(k,n)(\tau-\alpha(0,0))}\binom{\tau}{k}B_{n,k}(1!c_1,2!c_2,\dots)\right]\\ &= \sum_{k=0}^{n} \frac{k!}{n!}\left[\frac{2}{(an+bk+2)}\binom{an+bk+2}{k}B_{n,k}(1!c_1,2!c_2,\dots)\right]\\ &= 2\sum_{k=0}^{n}\binom{an+bk+1}{k-1}\frac{(k-1)!}{n!}B_{n,k}(1!c_1,2!c_2,\dots).\end{aligned}$$

For any fixed n, both sides of the claimed equation are polynomials in a and b. Since they coincide on an open subset of $\mathbb{R}^2$, they must coincide for all real numbers a and b. $\square$

Proof of Theorem 2.1. We proceed by induction in r. The case $r = 2$ was discussed in the previous lemma. Assume the formula (2.1) holds for products of length less than $r > 2$.

As before, we temporarily assume that both a and b are positive. For $n \geq 0$ we rewrite

$$\sum_{m_1+\cdots+m_{r-1}=n} y_{m_1}\cdots y_{m_{r-1}} = \sum_{k=0}^{n}\frac{r-1}{an+bk+r-1}\binom{an+bk+r-1}{k}\frac{k!}{n!}B_{n,k}(1!c_1,2!c_2,\dots).$$

Thus

$$\begin{aligned}\sum_{m_1+\cdots+m_r=n} y_{m_1}\cdots y_{m_r} &= \sum_{m=0}^{n} y_m \sum_{m_1+\cdots+m_{r-1}=n-m} y_{m_1}\cdots y_{m_{r-1}}\\ &= \sum_{m=0}^{n} y_m \sum_{j=0}^{n-m} \tfrac{r-1}{a(n-m)+bj+r-1}\tbinom{a(n-m)+bj+r-1}{j}\tfrac{j!}{(n-m)!}B_{n-m,j}.\end{aligned}$$

Writing y_m as in (2.2), we then get

$$\begin{aligned}&\frac{1}{r-1}\sum_{m_1+\cdots+m_r=n} y_{m_1}\cdots y_{m_r}\\ &\quad= \sum_{m=0}^{n}\left[\sum_{\ell=0}^{m}\frac{\binom{am+b\ell+1}{\ell}\ell!}{(am+b\ell+1)m!}B_{m,\ell}\right]\left[\sum_{j=0}^{n-m}\frac{\binom{a(n-m)+bj+r-1}{j}j!}{(a(n-m)+bj+r-1)(n-m)!}B_{n-m,j}\right]\\ &\quad= \sum_{m=0}^{n}\sum_{k=0}^{n}\sum_{\ell=0}^{k}\frac{\binom{a(n-m)+b(k-\ell)+r-1}{k-\ell}\binom{am+b\ell+1}{\ell}}{(am+b\ell+1)(a(n-m)+b(k-\ell)+r-1)}\frac{\ell!}{m!}\frac{(k-\ell)!}{(n-m)!}B_{m,\ell}B_{n-m,k-\ell}\\ &\quad= \sum_{k=0}^{n}\frac{k!}{n!}\left[\sum_{\ell=0}^{k}\sum_{m=\ell}^{n}\frac{\binom{\alpha(\ell,m)}{k-\ell}\binom{\tau-\alpha(\ell,m)}{\ell}\binom{n}{m}}{(\tau-\alpha(\ell,m))\alpha(\ell,m)\binom{k}{\ell}}B_{m,\ell}B_{n-m,k-\ell}\right]\end{aligned}$$

with $\alpha(\ell,m) = a(n-m)+b(k-\ell)+r-1$ and $\tau = an+bk+r$.

Finally, by Lemma 2.2,

$$\begin{aligned}\sum_{m_1+\cdots+m_r=n} y_{m_1}\cdots y_{m_r} &= (r-1)\sum_{k=0}^{n}\frac{k!}{n!}\left[\frac{\tau-\alpha(0,0)+\alpha(k,n)}{\tau\alpha(k,n)(\tau-\alpha(0,0))}\binom{\tau}{k}B_{n,k}(1!c_1,2!c_2,\dots)\right]\\ &= (r-1)\sum_{k=0}^{n}\frac{k!}{n!}\left[\frac{r\binom{an+bk+r}{k}}{(an+bk+r)(r-1)}B_{n,k}(1!c_1,2!c_2,\dots)\right]\\ &= r\sum_{k=0}^{n}\binom{an+bk+r-1}{k-1}\frac{(k-1)!}{n!}B_{n,k}(1!c_1,2!c_2,\dots).\end{aligned}$$

As in the previous lemma, this equation actually holds for all $a,b\in\mathbb{R}$ as claimed. □

3. Examples: Fibonacci, Tribonacci, Jacobsthal

As mentioned in the introduction, sequences of the form (1.1) with $a=0$ and $b=1$ can be used to describe linear recurrence sequences with constant coefficients. In this case, (1.1) takes the form

$$y_n=\sum_{k=0}^{n}\frac{k!}{n!}B_{n,k}(1!c_1,2!c_2,\dots)\text{ for } n\ge 0, \tag{3.1}$$

and the convolution formula (2.1) turns into

$$\begin{aligned}\sum_{m_1+\cdots+m_r=n} y_{m_1}\cdots y_{m_r} &= r\sum_{k=1}^{n}\binom{k+r-1}{k-1}\frac{(k-1)!}{n!}B_{n,k}(1!c_1,2!c_2,\dots)\\ &= \sum_{k=1}^{n}\binom{k+r-1}{k}\frac{k!}{n!}B_{n,k}(1!c_1,2!c_2,\dots).\end{aligned}$$

One can obtain (with a similar proof) the more general formula

$$\sum_{m_1+\cdots+m_r=n} y_{m_1-\delta}\cdots y_{m_r-\delta} = \sum_{k=0}^{n-\delta r}\binom{k+r-1}{k}\frac{k!}{(n-\delta r)!}B_{n-\delta r,k}(1!c_1,2!c_2,\dots)$$

for any integer $\delta\ge 0$, assuming $y_{-1}=y_{-2}=\cdots=y_{-\delta}=0$.

Example 3.1 (Fibonacci). *Consider the sequence defined by*

$$f_0=0,\ \ f_1=1,\ \ \textit{and}\ \ f_n=f_{n-1}+f_{n-2}\ \ \textit{for } n\ge 2.$$

Choosing $c_1=c_2=1$ and $c_j=0$ for $j\ge 3$ in (3.1)*, for $n\ge 1$ we have*

$$f_n=y_{n-1}=\sum_{k=0}^{n-1}\frac{k!}{(n-1)!}B_{n-1,k}(1,2,0,\dots)=\sum_{k=0}^{n-1}\binom{k}{n-1-k},$$

and

$$\sum_{m_1+\cdots+m_r=n} f_{m_1}\cdots f_{m_r}=\sum_{k=0}^{n-r}\binom{k+r-1}{k}\binom{k}{n-r-k}.$$

Example 3.2 (Tribonacci). *Let (t_n) be the sequence defined by*

$$t_0=t_1=0,\ \ t_2=1,\ \ \textit{and}\ \ t_n=t_{n-1}+t_{n-2}+t_{n-3}\ \textit{for } n\ge 3.$$

Choosing $c_1 = c_2 = c_3 = 1$ and $c_j = 0$ for $j \geq 4$ in (3.1), for $n \geq 2$ we have

$$t_n = y_{n-2} = \sum_{k=0}^{n-2} \frac{k!}{(n-2)!} B_{n-2,k}(1!, 2!, 3!, 0, \dots),$$

and since $B_{n,k}(1!, 2!, 3!, 0, \dots) = \frac{n!}{k!} \sum_{\ell=0}^{k} \binom{k}{k-\ell}\binom{k-\ell}{n+\ell-2k} = \frac{n!}{k!} \sum_{\ell=0}^{k} \binom{k}{\ell}\binom{\ell}{n-k-\ell}$, we get

$$t_n = \sum_{k=0}^{n-2} \sum_{\ell=0}^{k} \binom{k}{\ell}\binom{\ell}{n-2-k-\ell},$$

and

$$\sum_{m_1+\dots+m_r=n} t_{m_1} \cdots t_{m_r} = \sum_{k=0}^{n-2r} \sum_{\ell=0}^{k} \binom{k+r-1}{k}\binom{k}{\ell}\binom{\ell}{n-2r-k-\ell}.$$

Example 3.3 (Jacobsthal)**.** *The Jacobsthal polynomials are obtained by the recurrence*

$$J_0 = 0, \quad J_1 = 1, \text{ and}$$
$$J_n = J_{n-1} + 2xJ_{n-2} \text{ for } n \geq 2.$$

Choosing $c_1 = 1$, $c_2 = 2x$, and $c_j = 0$ for $j \geq 3$ in (3.1), for $n \geq 1$ we get

$$J_n = y_{n-1} = \sum_{k=0}^{n-1} \frac{k!}{(n-1)!} B_{n-1,k}(1, 2(2x), 0, \dots) = \sum_{k=0}^{n-1} \binom{k}{n-1-k}(2x)^{n-1-k},$$

and

$$\begin{aligned}\sum_{m_1+\dots+m_r=n} J_{m_1} \cdots J_{m_r} &= \sum_{k=0}^{n-r} \frac{k!}{(n-r)!}\binom{k+r-1}{k} B_{n-r,k}(1, 4x, 0, \dots) \\ &= \sum_{k=0}^{n-r} \binom{k+r-1}{k}\binom{k}{n-r-k}(2x)^{n-r-k}.\end{aligned}$$

4. Examples: Fuss–Catalan, Motzkin

All of the previous examples are related to the family (3.1). However, there are many other cases of interest. For example, let us consider the case when $a = 1$, $b = 0$, and $c_j = 0$ for $j \geq 3$. Since $B_{n,k}(c_1, 2c_2, 0, \dots) = \frac{n!}{k!}\binom{k}{n-k} c_1^{2k-n} c_2^{n-k}$, the family (1.1) can be written as

$$y_0 = 1, \quad y_n = \sum_{k=1}^{n} \frac{1}{k}\binom{n}{k-1}\binom{k}{n-k} c_1^{2k-n} c_2^{n-k} \text{ for } n \geq 1, \tag{4.1}$$

and the convolution formula (2.1) becomes

$$\sum_{m_1+\dots+m_r=n} y_{m_1} \cdots y_{m_r} = \sum_{k=1}^{n} \frac{r}{k}\binom{n+r-1}{k-1}\binom{k}{n-k} c_1^{2k-n} c_2^{n-k}. \tag{4.2}$$

Example 4.1 (Catalan). *If we let $c_1 = 2$ and $c_2 = 1$ in (4.1), for $n \geq 1$ we get*

$$\begin{aligned} y_n &= \sum_{k=1}^{n} \frac{1}{k} \binom{n}{k-1} \binom{k}{n-k} 2^{2k-n} \\ &= \frac{1}{n+1} \sum_{k=1}^{n} \binom{n+1}{k} \binom{k}{n-k} 2^{2k-n} \\ &= \frac{1}{n+1} \binom{2(n+1)}{n} = \frac{1}{n+2} \binom{2(n+1)}{n+1} = C_{n+1}. \end{aligned}$$

Here we used the identity

$$\sum_{k=\lceil \frac{n}{2} \rceil}^{n} \binom{x}{k} \binom{k}{n-k} 2^{2k} = 2^n \binom{2x}{n} \tag{4.3}$$

from Gould's collection [5, Identity (3.22)]. *As for convolutions,* (4.2) *leads to*

$$\begin{aligned} \sum_{m_1+\cdots+m_r=n} C_{m_1+1} \cdots C_{m_r+1} &= \sum_{k=1}^{n} \frac{r}{k} \binom{n+r-1}{k-1} \binom{k}{n-k} 2^{2k-n} \\ &= \frac{r}{n+r} \sum_{k=1}^{n} \binom{n+r}{k} \binom{k}{n-k} 2^{2k-n}. \end{aligned}$$

Using again (4.3)*, we arrive at the identity*

$$\sum_{m_1+\cdots+m_r=n} C_{m_1+1} \cdots C_{m_r+1} = \frac{r}{n+r} \binom{2(n+r)}{n}.$$

Example 4.2 (Motzkin). *Let us now consider* (4.1) *with $c_1 = 1$ and $c_2 = 1$. For $n \geq 1$,*

$$y_n = \sum_{k=1}^{n} \frac{1}{k} \binom{n}{k-1} \binom{k}{n-k} = \frac{1}{n+1} \sum_{k=1}^{n} \binom{n+1}{k} \binom{k}{n-k}.$$

These are the Motzkin numbers M_n. Moreover,

$$\sum_{m_1+\cdots+m_r=n} M_{m_1} \cdots M_{m_r} = \frac{r}{n+r} \sum_{k=0}^{n} \binom{n+r}{k} \binom{k}{n-k}.$$

We finish this section by considering the sequence (with $b \neq 0$):

$$y_0 = 1, \quad y_n = \sum_{k=1}^{n} \binom{bk}{k-1} \frac{(k-1)!}{n!} B_{n,k}(1!c_1, 2!c_2, \dots) \text{ for } n \geq 1.$$

Example 4.3 (Fuss–Catalan). *If $c_1 = 1$ and $c_j = 0$ for $j \geq 2$, then the above sequence becomes*

$$y_0 = 1, \quad y_n = \binom{bn}{n-1} \frac{(n-1)!}{n!} = \frac{1}{(b-1)n+1} \binom{bn}{n}.$$

Denoting $C_n^{(b)} = y_n$, and since $r\binom{bn+r-1}{n-1}\frac{(n-1)!}{n!} = \frac{r}{bn+r}\binom{bn+r}{n}$, we get the identity

$$\sum_{m_1+\cdots+m_r=n} C_{m_1}^{(b)} \cdots C_{m_r}^{(b)} = \frac{r}{bn+r} \binom{bn+r}{n}.$$

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