ALGEBRAIC STATEMENTS SIMILAR TO THOSE IN RAMANUJAN'S "LOST NOTEBOOK"

CURTIS COOPER

ABSTRACT. Ramanujan's "lost notebook" contains algebraic statements

$$\text{if } g^4 = 5, \text{ then } \frac{\sqrt[5]{3+2g} - \sqrt[5]{4-4g}}{\sqrt[5]{3+2g} + \sqrt[5]{4-4g}} = 2 + g + g^2 + g^3,$$

and

$$\text{if } g^5 = 2, \text{ then } \sqrt{1+g^2} = \frac{g^4 + g^3 + g - 1}{\sqrt{5}}.$$

In this paper we will discover algebraic statements similar to those in Ramanujan's "lost notebook". For example, we will prove algebraic statements like

$$\text{if } g^3 = 2, \text{ then } \frac{\sqrt[4]{111-87g} + \sqrt[4]{g-1}}{\sqrt[4]{111-87g} - \sqrt[4]{g-1}} = 2 + g + g^2,$$

and

$$\text{if } g^5 = 2, \text{ then } \sqrt{-3g^2 + 4g + 5} = g^4 - g^3 + g + 1.$$

1. INTRODUCTION

Page 344 of Ramanujan's "lost notebook" [2] contains twelve algebraic statements. Recently, Hirschhorn [1] gave simple proofs of these statements. Here are some of the statements.

If $g^5 = 3$, then

$$\frac{\sqrt{g^2+1} + \sqrt{5g-5}}{\sqrt{g^2+1} - \sqrt{5g-5}} = \frac{1}{g} + g + g^2 + g^3. \tag{1.1}$$

If $g^5 = 2$, then

$$\sqrt{1+g^2} = \frac{g^4 + g^3 + g - 1}{\sqrt{5}}. \tag{1.2}$$

If $g^5 = 2$, then

$$\sqrt{4g-3} = \frac{g^9 + g^7 - g^6 - 1}{\sqrt{5}}. \tag{1.3}$$

If $g^5 = 2$, then

$$\sqrt[5]{1+g+g^3} = \frac{\sqrt{1+g^2}}{\sqrt[10]{5}}, \tag{1.4}$$

If $g^4 = 5$, then

$$\frac{\sqrt[5]{3+2g} - \sqrt[5]{4-4g}}{\sqrt[5]{3+2g} + \sqrt[5]{4-4g}} = 2 + g + g^2 + g^3. \tag{1.5}$$

In this paper we will discover algebraic statements similar to the above statements in Ramanujan's "lost notebook". The paper is organized as follows. Section 2 gives algebraic statements similar to (1.1). Section 3 gives algebraic statements similar to (1.2) and (1.3).

Section 4 gives algebraic statements similar to (1.4) and Section 5 gives algebraic statements similar to (1.5). Finally, Section 6 gives other algebraic statements in the spirit of Ramanujan.

One technique we will use throughout the paper is *componendo et dividendo*, which states that

$$\frac{a+b}{a-b} = \frac{c+d}{c-d} \text{ if and only if } \frac{a}{b} = \frac{c}{d}.$$

2. Algebraic Statement Similar to (1.1)

We wish to find an algebraic statement similar to Ramanujan's (1.1). The following theorem is similar to (1.1).

Theorem 2.1. *If $g^5 = 2$, then*

$$\frac{\sqrt{4g^2+g+2}+\sqrt{8g^2+41g-54}}{\sqrt{4g^2+g+2}-\sqrt{8g^2+41g-54}} = \frac{1}{g}+g+g^2+g^3. \tag{2.1}$$

Proof. Equation (2.1) can be written as

$$\frac{\sqrt{4g^2+g+2}+\sqrt{8g^2+41g-54}}{\sqrt{4g^2+g+2}-\sqrt{8g^2+41g-54}} = \frac{1+g^2+g^3+g^4}{g}.$$

Thus, by *componendo et dividendo*, we need to show that

$$\sqrt{\frac{4g^2+g+2}{8g^2+41g-54}} = \frac{1+g+g^2+g^3+g^4}{1-g+g^2+g^3+g^4}.$$

This is equivalent to showing that

$$(1+g+g^2+g^3+g^4)^2(8g^2+41g-54) = (1-g+g^2+g^3+g^4)^2(4g^2+g+2). \tag{2.2}$$

Expanding both sides of (2.2) and using the fact that $g^5 = 2$, the left- and right-hand sides of (2.2) are equal and the theorem is proved. □

3. Algebraic Statements Similar to (1.2) and (1.3)

We wish to find an algebraic statement similar to Ramanujan's (1.2) and (1.3). The following theorem is similar to (1.2).

Theorem 3.1. *If $g^5 = 8$, then*

$$\sqrt{2g^2-3} = \frac{g^4+2g^3-2g^2-2}{2\sqrt{5}}.$$

Proof. Using the fact that $g^5 = 8$, we have the following equalities.

$$\begin{aligned}(g^4+2g^3-2g^2-2)^2 &= g^8+4g^7-8g^5-8g^3+8g^2+4\\ &= 8g^3+32g^2-64-8g^3+8g^2+4\\ &= 40g^2-60 = 20(2g^2-3).\end{aligned}$$

This proves the theorem. □

To find more algebraic identities similar to (1.2), we wrote a C++ program to search for solutions to

$$(R+Sg+Tg^2+Ug^3+Vg^4)^2 = Cg^2+E.$$

We discovered the following (two) theorems.

Theorem 3.2. *If* $g^5 = 18$*, then*

$$\sqrt{g^2-3} = \frac{g^4+3g^3-6g^2-3g-9}{15}.$$

Proof.

$$\begin{aligned}(g^4+3g^3-6g^2-3g-9)^2 &= g^8+6g^7-3g^6-42g^5-18g^3+117g^2+54g+81\\ &= 18g^3+108g^2-54g-756-18g^3+117g^2+54g+81\\ &= 225g^2-675 = 225(g^2-3).\end{aligned}$$

□

Theorem 3.3. *If* $g^5 = 49$*, then*

$$\sqrt{8g^2-7} = \frac{6g^4+14g^3-14g^2+14g-49}{35}.$$

Proof.

$$\begin{aligned}&(6g^4+14g^3-14g^2+14g-49)^2\\ &= 36g^8+168g^7+28g^6-224g^5-1764g^3+1568g^2-1372g+2401\\ &= 1764g^3+8232g^2+1372g-10976-1764g^3+1568g^2-1372g+2401\\ &= 9800g^2-8575 = 1225(8g^2-7).\end{aligned}$$

□

The following theorem is similar to Ramanujan's (1.3).

Theorem 3.4. *If* $g^5 = 8$*, then*

$$\sqrt{g+2} = \frac{g^4-g^3+4g+4}{2\sqrt{10}}.$$

Proof. Using the fact that $g^5 = 8$, we have the following equalities.

$$\begin{aligned}(g^4-g^3+4g+4)^2 &= g^8-2g^7+g^6+8g^5-8g^3+16g^2+32g+16\\ &= 8g^3-16g^2+8g+64-8g^3+16g^2+32g+16\\ &= 40g+80 = 40(g+2).\end{aligned}$$

This proves the theorem. □

To find more algebraic identities similar to (1.3), we wrote a C++ program to search for solutions to

$$(R+Sg+Tg^2+Ug^3+Vg^4)^2 = Dg+E.$$

We discovered the following (three) theorems. The proofs are similar to the proof above.

Theorem 3.5. *If* $g^5 = 12$*, then*

$$\sqrt{11g-7} = \frac{g^4-g^3+2g^2-8g-10}{2\sqrt{5}}.$$

Theorem 3.6. *If $g^5 = 2$, then*
$$\sqrt{4g-3} = \frac{2g^4 + 2g^2 - 2g - 1}{\sqrt{5}}.$$

Theorem 3.7. *If $g^5 = 4$, then*
$$\sqrt{g+1} = \frac{g^4 + g^3 + 2g^2 - 2}{2\sqrt{5}}.$$

Theorem 3.8. *If $g^5 = 7$, then*
$$\sqrt{-g+8} = \frac{2g^4 - g^3 - 2g^2 + 6g + 2}{5}.$$

Theorem 3.9. *If $g^5 = 24$, then*
$$\sqrt{-g+2} = \frac{g^4 - g^3 - 4g^2 + 4g - 4}{10\sqrt{2}}.$$

We discovered the following (two) theorems similar to Ramanujan's (1.2) and (1.3).

Theorem 3.10. *If $g^5 = 2$, then*
$$\sqrt{8g^2 - 20g + 17} = g^9 - g^7 + g^6 - 1.$$

Proof. Using the fact that $g^5 = 2$, we have the following equalities.
$$\begin{aligned}(g^9 - g^7 + g^6 - 1)^2 &= (2g^4 - 2g^2 + 2g - 1)^2 \\ &= 4g^8 - 8g^6 + 8g^5 - 8g^3 + 8g^2 - 4g + 1 \\ &= 8g^3 - 16g + 16 - 8g^3 + 8g^2 - 4g + 1 \\ &= 8g^2 - 20g + 17.\end{aligned}$$

This proves the theorem. □

Theorem 3.11. *If $g^5 = 2$, then*
$$\sqrt{-3g^2 + 4g + 5} = g^4 - g^3 + g + 1.$$

4. Algebraic Statements Similar to (1.4)

We wish to find an algebraic statement similar to Ramanujan's (1.4). The following theorem is similar to (1.4).

Theorem 4.1. *If $g^5 = 8$, then*
$$\sqrt[5]{2 + 2g + g^2} = \frac{\sqrt{2+g}}{\sqrt[10]{10}}.$$

Proof. Using the fact that $g^5 = 8$, we have the following equalities.
$$\begin{aligned}(2+g)^5 &= 32 + 80g + 80g^2 + 40g^3 + 10g^4 + g^5 \\ &= 32 + 80g + 80g^2 + 40g^3 + 10g^4 + 8 \\ &= 40 + 80g + 80g^2 + 40g^3 + 10g^4 \\ &= 10(4 + 8g + 8g^2 + 4g^3 + g^4) \\ &= 10(2 + 2g + g^2)^2.\end{aligned}$$

This proves the theorem. □

To find more algebraic identities similar to (1.4), we wrote a C++ program to search for solutions to

$$(R+Sg+Tg^2)^5 = (A+Bg+Cg^2+Dg^3)^2.$$

The next theorem is one that we discovered. The proof is similar to the proof above.

Theorem 4.2. *If $g^5 = 8$, then*

$$\sqrt[5]{1+g} = \frac{\sqrt{2+g}}{\sqrt[10]{40}}.$$

5. Algebraic Statements Similar to (1.5)

We wish to find an algebraic statement similar to Ramanujan's (1.5). To construct a result similar to (1.5), we want to find integers h, A, B, C, D, and E such that if $g^4 = h+1$, then

$$\sqrt[5]{\frac{Ag+B}{Cg+D}} = \frac{E+1+g+g^2+g^3}{1+g+g^2+g^3}. \tag{5.1}$$

Then, by *componendo et dividendo*, we would have

$$\frac{\sqrt[5]{Ag+B}+\sqrt[5]{Cg+D}}{\sqrt[5]{Ag+B}-\sqrt[5]{Cg+D}} = \frac{E+2+2g+2g^2+2g^3}{E}.$$

Simplifying the RHS of (5.1) and using the fact that $g^4 = h+1$, we have

$$\begin{aligned}\sqrt[5]{\frac{Ag+B}{Cg+D}} &= \frac{E+1+g+g^2+g^3}{1+g+g^2+g^3}\\ &= \frac{E+\frac{h}{g-1}}{\frac{h}{g-1}} = \frac{Eg+h-E}{h}.\end{aligned}$$

Thus,

$$\frac{Ag+B}{Cg+D} = \frac{(Eg+h-E)^5}{h^5}$$

and so

$$h^5(Ag+B) = (Cg+D)(Eg+h-E)^5. \tag{5.2}$$

Expanding the polynomial on the RHS of (5.2), we have

$$\begin{aligned}&CE^5g^6 + (DE^5 - 5CE^5 + 5ChE^4)g^5\\ &+ (10Ch^2E^3 + 5DhE^4 - 20ChE^4 + 10CE^5 - 5DE^5)g^4\\ &+ (10Ch^3E^2 + 30ChE^4 + 10DE^5 + 10Dh^2E^3 - 30Ch^2E^3 - 10CE^5 - 20DhE^4)g^3\\ &+ (5Ch^4E - 20ChE^4 - 30Dh^2E^3 - 10DE^5 + 30Ch^2E^3 - 20Ch^3E^2 + 5CE^5\\ &\qquad + 30DhE^4 + 10Dh^3E^2)g^2\\ &+ (-10Ch^2E^3 + 10Ch^3E^2 + 5Dh^4E - 20Dh^3E^2 + 30Dh^2E^2 + 30Dh^2E^3 + 5DE^5\\ &\qquad + 5ChE^4 - 20DhE^4 - 5Ch^4E - CE^5 + Ch^5)g\\ &- 10Dh^2E^3 - DE^5 - 5Dh^4E + Dh^5 + 5DhE^4 + 10Dh^3E^2.\end{aligned}$$

Assuming $g^4 = h+1$, we continue to simplify the polynomial on the RHS of (5.2) by defining

$$\begin{aligned}
&CE^5g^6 = CE^5(h+1)g^2 = Fg^2\\
&(DE^5 - 5CE^5 + 5ChE^4)g^5 = (DE^5 - 5CE^5 + 5ChE^4)(h+1)g = Gg\\
&(10Ch^2E^3 + 5DhE^4 - 20ChE^4 + 10CE^5 - 5DE^5)g^4\\
&\qquad = (10Ch^2E^3 + 5DhE^4 - 20ChE^4 + 10CE^5 - 5DE^5)(h+1) = H.
\end{aligned}$$

In addition we define

$$\begin{aligned}
&10Ch^3E^2 + 30ChE^4 + 10DE^5 + 10Dh^2E^3 - 30Ch^2E^3\\
&\qquad - 10CE^5 - 20DhE^4 = I\\
&5Ch^4E - 20ChE^4 - 30Dh^2E^3 - 10DE^5 + 30Ch^2E^3 - 20Ch^3E^2\\
&\qquad + 5CE^5 + 30DhE^4 + 10Dh^3E^2 = J\\
&-10Ch^2E^3 + 10Ch^3E^2 + 5Dh^4E - 20Dh^3E^2 + 30Dh^2E^2 + 30Dh^2E^3\\
&\qquad + 5DE^5 + 5ChE^4 - 20DhE^4 - 5Ch^4E - CE^5 + Ch^5 = K\\
&-10Dh^2E^3 - DE^5 - 5Dh^4E + Dh^5 + 5DhE^4 + 10Dh^3E^2 = L.
\end{aligned}$$

Thus, the expanded polynomial on the RHS of (5.2) is

$$Ig^3 + (F+J)g^2 + (G+K)g + (H+L).$$

To simplify this polynomial, we want $I = 0$, $F = -J$, $h \neq 0$, $h \neq -1$, $C \neq 0$, and $E \neq 0$. Equation (5.2) becomes

$$Ah^5g + Bh^5 = (G+K)g + (H+L).$$

Therefore, we wrote a C++ program to search for integers C, D, E, and h with the above constraints and with A and B integers. We found the following solutions.

C	D	E	h	A	B
4	-4	2	4	2	3
1	2	-79	79	512	-1536
4	12	-202	404	486	-2187

The first line of the table is Ramanujan's algebraic statement (1.5). Here are the other two theorems.

Theorem 5.1. *If $g^4 = 80$, then*

$$\frac{\sqrt[5]{512g - 1536} + \sqrt[5]{g+2}}{\sqrt[5]{512g - 1536} - \sqrt[5]{g+2}} = \frac{77 - 2g - 2g^2 - 2g^3}{79}. \tag{5.3}$$

Theorem 5.2. *If $g^4 = 405$, then*

$$\frac{\sqrt[5]{486g - 2187} + \sqrt[5]{4g+12}}{\sqrt[5]{486g - 2187} - \sqrt[5]{4g+12}} = \frac{200 - 2g - 2g^2 - 2g^3}{202}.$$

But, Theorems 5.1 and 5.2 are equivalent to Ramanujan's equation (1.5). To see this, we start by rewriting Ramanujan's equation (1.5), using some algebra, as

$$\frac{\sqrt[5]{2g+3} + \sqrt[5]{4g-4}}{\sqrt[5]{2g+3} - \sqrt[5]{4g-4}} = 2 + g + g^2 + g^3. \tag{5.4}$$

Changing equation (5.4) using *componendo et dividendo* results in the following equation.

$$\sqrt[5]{\frac{2g+3}{4g-4}} = \frac{3+g+g^2+g^3}{1+g+g^2+g^3}. \tag{5.5}$$

Changing equation (5.3) using *componendo et dividendo* results in the following algebraic equation.

$$\sqrt[5]{\frac{512g-1536}{g+2}} = \frac{-78+g+g^2+g^3}{1+g+g^2+g^3}. \tag{5.6}$$

Now we show equation (5.5) and equation (5.6) are equivalent. Start with equation (5.6) under the assumption that $g^4 = 80$. Substituting $g = -2f$ into equation (5.6), we have the following equation under the assumption that $f^4 = 5$.

$$\sqrt[5]{\frac{-1024f-1536}{-2f+2}} = \frac{-78-2f+4f^2-8f^3}{1-2f+4f^2-8f^3}. \tag{5.7}$$

Simplifying equation (5.7), we obtain

$$\sqrt[5]{\frac{2f+3}{4f-4}} = \frac{-78-2f+4f^2-8f^3}{4-8f+16f^2-32f^3}. \tag{5.8}$$

But, if $f^4 = 5$, we have that

$$\frac{-78-2f+4f^2-8f^3}{4-8f+16f^2-32f^3} = \frac{3+f+f^2+f^3}{1+f+f^2+f^3}. \tag{5.9}$$

We can prove this by showing that if $f^4 = 5$, then

$$(-78-2f+4f^2-8f^3)(1+f+f^2+f^3) = (4-8f+16f^2-32f^3)(3+f+f^2+f^3).$$

But equations (5.8) and (5.9) produce equation (5.5). Thus, we have shown that Theorem 5.1 is equivalent to Ramanujan's equation (1.5). Following the same procedure with $g = -3f$ shows that Theorem 5.2 is equivalent to Ramanujan's equation (1.5).

6. More Algebraic Statements

We state and prove some theorems that are similar to some of Ramanujan's algebraic statements.

Theorem 6.1. *If $g^3 = 2$, then*

$$\frac{\sqrt[4]{111-87g}+\sqrt[4]{g-1}}{\sqrt[4]{111-87g}-\sqrt[4]{g-1}} = 2+g+g^2. \tag{6.1}$$

Proof. Using *componendo et dividendo* and the fact that

$$1 = g^3 - 1 = (1+g+g^2)(g-1)$$

we rewrite (6.1) as

$$\sqrt[4]{\frac{111-87g}{g-1}} = \frac{3+g+g^2}{1+g+g^2} = \frac{1+g+g^2+2}{1+g+g^2}$$
$$= \frac{\frac{1}{g-1}+2}{\frac{1}{g-1}} = 2g-1.$$

But, since $g^3 = 2$, we have that

$$\begin{aligned}(2g-1)^4 &= (2g)^4 - 4(2g)^3 + 6(2g)^2 - 4(2g) + 1 = 16g^4 - 32g^3 + 24g^2 - 8g + 1\\ &= 32g - 64 + 24g^2 - 8g + 1 = 24g^2 + 24g - 63\\ &= 24g^2 + 24g + 24 - 87 = \frac{24}{g-1} - 87 = \frac{111-87g}{g-1}.\end{aligned}$$

□

We can generalize this result in the following way.

Theorem 6.2. *Let A and h be given integers and let*

$$\begin{aligned}B &= 6A^2(h-A)^2\\ D &= A^4(h+1) + 4A(h-A)^3\\ C &= 4A^3(h-A)(h+1) + (h-A)^4.\end{aligned}$$

Then if $B = D$ and $B \neq 0$, we have the following result. If $g^3 = h+1$, then

$$\pm\sqrt[4]{\frac{(C-B)g + Bh + B - C}{h^4 g - h^4}} = \frac{A+1+g+g^2}{1+g+g^2}. \tag{6.2}$$

We will choose the plus or minus sign depending on the real value of the RHS of equation (6.2).

Proof. We wish to find an unknown function f of A and h ($g^3 = h+1$) such that

$$\pm\sqrt[4]{f} = \frac{A+1+g+g^2}{1+g+g^2} = \frac{\frac{h}{g-1} + A}{\frac{h}{g-1}} = \frac{1}{h}\left(Ag + h - A\right).$$

So,

$$\begin{aligned}f &= \frac{1}{h^4}\left(Ag + h - A\right)^4\\ &= \frac{1}{h^4}\left(A^4g^4 + 4A^3g^3(h-A) + 6A^2g^2(h-A)^2 + 4Ag(h-A)^3 + (h-A)^4\right)\\ &= \frac{1}{h^4}\left(A^4(h+1)g + 4A^3(h+1)(h-A) + 6A^2(h-A)^2g^2 + 4A(h-A)^3g + (h-A)^4\right)\\ &= \frac{1}{h^4}\left(6A^2(h-A)^2g^2 + \left(A^4(h+1) + 4A(h-A)^3\right)g + 4A^3(h+1)(h-A) + (h-A)^4\right).\end{aligned}$$

Now the last expression for f is equal to

$$\begin{aligned}f &= \frac{1}{h^4}\left(Bg^2 + Bg + C\right) = \frac{1}{h^4}\left(B(g^2+g+1) + C - B\right)\\ &= \frac{1}{h^4}\left(\frac{Bh}{g-1} + C - B\right)\\ &= \frac{1}{h^4}\left(\frac{Bh + (C-B)g + B - C}{g-1}\right)\\ &= \frac{1}{h^4}\frac{(C-B)g + Bh + B - C}{g-1}.\end{aligned}$$

So

$$\pm\sqrt[4]{f} = \sqrt[4]{\frac{(C-B)g + Bh + B - C}{h^4(g-1)}} = \frac{A+1+g+g^2}{1+g+g^2}.$$

□

The following table gives A and h which apply to the theorem. We omitted the values of h where $h+1$ is a perfect cube. The range of A and h which was searched is -30000 to 30000 for both variables.

A	h	B	C
-16384	-24576	108086391056891904	148618787703226368
-8192	-12288	6755399441055744	9288674231451648
-511	4599	40910505984600	-11864046735534000
-188	846	226729497984	-22134467240688
-185	-1665	449798640000	-57574225920000
-117	351	17989317216	-1007401764096
-55	55	219615000	-3953070000
-22	-99	17217816	-286246191
2	1	24	-63
8192	12288	6755399441055744	9288674231451648
16384	24576	108086391056891904	148618787703226368

The third line from the bottom of the table is Theorem 6.1. If we construct an algebraic statement from the fifth line from the bottom of the table with $A = -55$, $h = 55$, $B = 219615000$ and $C = -3953070000$, we obtain the following theorem.

Theorem 6.3. *If $g^3 = 56$, then*

$$\frac{\sqrt[4]{1776-456g} + \sqrt[4]{g-1}}{\sqrt[4]{1776-456g} - \sqrt[4]{g-1}} = \frac{55}{53-2g-2g^2}. \tag{6.3}$$

However, this statement can be simplified to the following statement.

Theorem 6.4. *If $h^3 = 7$, then*

$$\frac{\sqrt[4]{111-57h} + \sqrt[4]{2h-1}}{\sqrt[4]{111-57h} - \sqrt[4]{2h-1}} = \frac{7+h+h^2}{5-h-h^2}. \tag{6.4}$$

To prove these two theorems are equivalent, we will show that their alternate forms using *componendo et dividendo* are equivalent. The alternate form of Theorem 6.3 is if $g^3 = 56$, then

$$\sqrt[4]{\frac{1776-456g}{g-1}} = \frac{54-g-g^2}{1+g+g^2} \tag{6.5}$$

and the alternate form of Theorem 6.4 is if $h^3 = 7$, then

$$\sqrt[4]{\frac{111-57h}{2h-1}} = \frac{6}{1+h+h^2}. \tag{6.6}$$

We start with equation (6.5). Substituting $g = 2h$ in equation (6.5) and simplifying, we obtain the following algebraic statement.

If $h^3 = 7$, then
$$\sqrt[4]{\frac{111-57h}{2h-1}} = \frac{27-h-2h^2}{1+2h+4h^2}. \tag{6.7}$$
Finally, if $h^3 = 7$, we have
$$\frac{27-h-2h^2}{1+2h+4h^2} = \frac{6}{1+h+h^2}. \tag{6.8}$$
But equations (6.7) and (6.8) give equation (6.6). Therefore, since their alternate statements are equivalent, Theorem 6.3 and Theorem 6.4 are equivalent.

We give another algebraic statement theorem and its proof.

Theorem 6.5. *If $g^5 = 4$, then*
$$\frac{3\sqrt{3g^2+4g+6}+\sqrt{55g^2+40g-50}}{3\sqrt{3g^2+4g+6}-\sqrt{55g^2+40g-50}} = \frac{6+g^2-g^3}{-g^2+g^3}. \tag{6.9}$$
Proof. Using the equality
$$\frac{6+g^2-g^3}{-g^2+g^3} = \frac{3+3+g^2-g^3}{3-3-g^2+g^3},$$
we rewrite (6.9) using *componendo et dividendo* as
$$3\sqrt{\frac{3g^2+4g+6}{55g^2+40g-50}} = \frac{3}{3-g^2+g^3}.$$
But, since $g^5 = 4$, we have that
$$\begin{aligned}&(3g^2+4g+6)(3+g^2-g^3)^2\\ &= 54+36g+63g^2-12g^3-26g^5+g^6-2g^7+3g^8\\ &= 54+36g+63g^2-12g^3-104+4g-8g^2+12g^3\\ &= 55g^2+40g-50\end{aligned}$$
and the theorem is proved. □

Here is another algebraic statement and its proof.

Theorem 6.6. *If $g^5 = 2$, then*
$$\frac{\sqrt[3]{5g^2+1}+\sqrt[3]{35g^2+g-43}}{\sqrt[3]{5g^2+1}-\sqrt[3]{35g^2+g-43}} = \frac{2+g-g^2}{-g+g^2}. \tag{6.10}$$
Proof. Using the equality
$$\frac{2+g-g^2}{-g+g^2} = \frac{1+1+g-g^2}{1-1-g+g^2},$$
we rewrite (6.10) using *componendo et dividendo* as
$$\sqrt[3]{\frac{5g^2+1}{35g^2+g-43}} = \frac{1}{1+g-g^2}.$$

But, since $g^5 = 2$, we have that

$$\begin{aligned}
&(1+g-g^2)^3(5g^2+1) \\
&= -5g^8 + 15g^7 - g^6 - 22g^5 + 10g^3 + 5g^2 + 3g + 1 \\
&= -10g^3 + 30g^2 - 2g - 44 + 10g^3 + 5g^2 + 3g + 1 \\
&= 35g^2 + g - 43
\end{aligned}$$

and the theorem is proved. □

7. Acknowledgement

The author wishes to thank an anonymous referee for the very helpful suggestions which improved the paper.

References

[1] M. D. Hirschhorn and V. Sinescu, *Elementary algebra in Ramanujan's lost notebook*, The Fibonacci Quarterly, **51.2** (2013), 123–129.

[2] S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, New Delhi, Narosa, 1988, p. 344.

MSC2010: 11A55

Department of Mathematics and Computer Science, University of Central Missouri, Warrensburg, Mo 64093

E-mail address: cooper@ucmo.edu