A CAYLEY-HAMILTON AND CIRCULANT APPROACH TO JUMP SUMS

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Abstract. Define the Pascal Triangle jump sum by $\begin{bmatrix} n \\ k \end{bmatrix}_j = \sum_{m \equiv k\ (j)} \binom{n}{m}$, with $m \equiv k\ (j)$ meaning, as usual, $m \equiv k \pmod{j}$, and with with $\binom{n}{m} = 0$, if either $m < 0$ or $m > n$. The jump sum function adds every j-th entry in the n-th row of Pascal's Triangle starting at column k. The jump sum has been studied by several authors over the last 2 decades. Both recursions and explicit formulae have been given as well as several interesting number-theoretic applications. Varied proof methods have been presented including inductive, combinatoric, generating-function, and Riordan-array proofs. The goal of this paper is to provide an extremely compact proof of the recursions satisfied by the jump-sum functions using (i) the theory of circulant matrices and (ii) an extension of the Cayley-Hamilton Theorem that studies the values of a polynomial - whose zeroes are some, but not all, eigenvalues of a matrix - evaluated at that matrix. This matrix approach allows us to derive closed functional forms for some coefficients in the recursions.

1. Introduction

Define the (Pascal Triangle) jump-sum by

$$\begin{bmatrix} n \\ k \end{bmatrix}_j = \sum_{m \equiv k\ (n)} \binom{n}{m}, \tag{1.1}$$

with $m \equiv k\ (n)$ meaning, as usual, $m \equiv k \pmod{n}$, and with $\binom{n}{m} = 0$, if either $m < 0$ or $m > n$. The jump sum function adds every j-th entry in the n-th row of Pascal's Triangle, the summation process beginning at column k. Note, that although if say $k < 0$ that $\binom{n}{k} = 0$, nevertheless, $\begin{bmatrix} n \\ k \end{bmatrix}_j \neq 0$, since the value of $\begin{bmatrix} n \\ k \end{bmatrix}_j$ depends on the congruence class of k modulo j.

The jump-sums satisfy recursions and in fact, they "can be expressed in terms of some linearly recurrent sequences with orders bounded by $\phi(j)/2$," [19]. See also [20, 3].

Varied applications of the jump-sums exist including, values of Bernoulli and Euler polynomials at rational points [6, 20], values of quadratic characters [19, 13], as well as derivation of interesting new congruences for primes and various number theoretic quotients [16, 19].

Explicit formulas for $\begin{bmatrix} n \\ k \end{bmatrix}_j$ for $j = 3, 4, 5, 8, 10, 12$ may be found in [3, 15, 14, 16, 19].

A variety of proof methods have been applied including proofs by combinatorics [1], Riordan-arrays [10], and generating functions [12], as well as Jensen [2] and WZ proofs[5]. In this paper, we present a very compact proof based on the theory of circulants and extensions of the Cayley-Hamilton Theorem to the case where the factors of a polynomial contain some, but not all, of the eigenvalues of a matrix, and that polynomial is evaluated at that matrix.

To motivate our approach, we first review in Table 1 some numerical values of $\begin{bmatrix} 3 \\ k \end{bmatrix}_3$. Table 2 presents numerical values of $3\begin{bmatrix} n \\ k \end{bmatrix}_3 - 2^n$. Values of Table 2 can easily be computed from corresponding values in Table 1. Rows 3,6, and 9 of Table 2, suggest that the value of $3\begin{bmatrix} 3n \\ k \end{bmatrix}_3 -$

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n	Pascal Triangle Row	$k=0$	$k=1$	$k=2$
3	1 3 3 1	2	3	3
4	1 4 6 4 1	5	5	6
5	1 5 10 10 5 1	11	10	11
6	1 6 15 20 15 6 1	22	21	21
7	1 7 21 35 35 21 7 1	43	43	42
8	1 8 28 56 70 56 28 8 1	85	86	85
9	1 9 36 84 126 126 84 36 9 1	170	171	171

TABLE 1. Values of $\begin{bmatrix} n \\ k \end{bmatrix}_3$ based on (1.1), for small n.

n	Pascal Row	$k=0$	$k=1$	$k=2$
3	1 3 3 1	-2	1	1
4	1 4 6 4 1	-1	-1	2
5	1 5 10 10 5 1	1	-2	1
6	1 6 15 20 15 6 1	2	-1	-1
7	1 7 21 35 35 21 7 1	1	1	-2
8	1 8 28 56 70 56 28 8 1	-1	2	-1
9	1 9 36 84 126 126 84 36 9 1	-2	1	1

TABLE 2. Values of $3\begin{bmatrix} n \\ k \end{bmatrix}_3 - 2^n$ for small n. The corresponding values of $\begin{bmatrix} n \\ k \end{bmatrix}_3$ may be found in Table 1.

$n \equiv r\ (3)$	$k=0$	$k=1$	$k=2$
$r=0$	$2(-1)^n$	$-1(-1)^n$	$-1(-1)^n$
$r=1$	$-1(-1)^n$	$-1(-1)^n$	$2(-1)^n$
$r=2$	$-1(-1)^n$	$2(-1)^n$	$-1(-1)^n$

TABLE 3. Values of $c(k,n) = 3\begin{bmatrix} 3n \\ k \end{bmatrix}_3 - 2^{3n}$ based on the congruence class of n and k modulo 3.

$2^{3n} = c_3(k,n)$ depends only on the congruence class modulo 3 of n and k. An elementary proof based on the Pascal Recursion is presented in [3]. Table 3 presents all values of $c_3(k,n)$. Closed functional forms for $j\begin{bmatrix} jn \\ k \end{bmatrix}_j - 2^{jn} = c_j(k,n)$, depending only on the congruence class of k and n modulo j, have been computed for $j = 4, 5, 8, 10, 12$ [20, 3].

A further study of either Table 2 or Table 3 shows that for each fixed k and $l \in \{0,1,2\}$ the sequence $\{3\begin{bmatrix} 3n+l \\ k \end{bmatrix}_3 - 2^{3n+l}\}_{n \geq 1}$ satisfies the recursion $G_n + G_{n-1} = 0$.

We can exploit this uniformity to obtain a new approach to the jump-sum recursions based on matrices and vectors. Fix j and l with $0 \leq l \leq j-1$. Define the vector

$$G_n^{(j,l)} = G_n = \langle j\begin{bmatrix} jn+l \\ 0 \end{bmatrix}_j - 2^{jn+l}, j\begin{bmatrix} jn+l \\ 1 \end{bmatrix}_j - 2^{jn+l}, \dots, j\begin{bmatrix} jn+l \\ k-1 \end{bmatrix}_j - 2^{jn+l}\rangle. \tag{1.2}$$

n	$3n+0$	$G_n^{(3,0)}$	$3n+1$	$G_n^{(3,1)}$	$3n+2$	$G_n^{(3,2)}$
1	3	$\langle -2,1,1\rangle$	4	$\langle -1,-1,2\rangle$	5	$\langle 1,-2,1\rangle$
2	6	$\langle 2,-1,-1\rangle$	7	$\langle 1,1,-2\rangle$	8	$\langle -1,2,-1\rangle$
3	9	$\langle -2,1,1\rangle$	10	$\langle -1,-1,2\rangle$	11	$\langle 1,-2,1\rangle$
4	12	$\langle 2,-1,-1\rangle$	13	$\langle 1,1,-2\rangle$	14	$\langle -1,2,-1\rangle$

TABLE 4. Values of $G_n^{(3,l)}, 0 \leq l \leq 2$, based on (1.2), with the values of the vector components previously computed in Tables 1-3. We have for all n and l, $G_n + G_{n-1} = 0$.

In the rest of the paper, we may notationally indicate such vectors by combining set notation with angle brackets as follows.

$$G_n^{(j,l)} = G_n = \langle j \begin{bmatrix} jn+l \\ k \end{bmatrix}_j - 2^{jn+l} : 0 \leq k \leq j-1\rangle.$$

When using such a notation, the angle brackets indicate that we are regarding the elements of the underlying set as ordered (that is, they are a vector). As an example of our notation, $\langle t : 3 \geq t \geq 1\rangle = \langle 3,2,1\rangle$ while $\langle t : 1 \leq t \leq 3\rangle = \langle 1,2,3\rangle$.

Table 4 shows values of G_n for initial values of n for $j = 3$ and for all congruence classes of l modulo j. As can be seen, the vectors $\{G_n\}_{n\geq 1}$ satisfy the vector recursion $G_n + G_{n-1} = 0$, uniformly for all l.

The relationship between G_n and G_{n-1} can be described using a matrix. To do this we first recall that $\mathbf{Circ}(a_0, a_1, a_2, \ldots, a_{m-1})$ is the $m \times m$ matrix, Q, whose first row is $a_0, a_1, a_2, \ldots, a_{m-1}$ with $Q_{i,l} = Q_{i',l'}$ if $l - i \equiv l' - i'$ (m) [4].

Throughout the paper, the matrix entry in the x-th row and y-th column of a matrix Q will be denoted by either $Q(x,y)$ or $Q_{x,y}$. Similarly $v(k)$ or v_k will indicate the k-th component of the vector v. The notation $Q_{*,y}$ or $Q_{x,*}$ will indicate the y-th column or x-th row respectively. The notation Q_j (with one subscript) indicates Q evaluated at parameter j. We abuse vector notation so that e.g. the vector v in Qv is perceived as a column vector even though originally defined as a row vector.

Using these notations, we define

$$M_j = \mathbf{Circ}(\binom{j}{0} + \binom{j}{j}, \binom{j}{1}, \binom{j}{2}, \ldots, \binom{j}{j-1}). \tag{1.3}$$

Matrix M_j is closely related to the circulant matrix underlying Wendt's determinant [7, 21]. In fact, $W_j = \mathbf{Det}(M_j - I_j)$, where I_j is the $j \times j$ identity matrix. However, this fact will not be further used in this paper.

Proposition 1.1. *For any fixed* $l, 0 \leq l \leq j-1$, *and for all* $n \geq 1$,

$$M_j G_n^{(j,l)} = G_{n+1}^{(j,l)}. \tag{1.4}$$

.

Throughout the rest of the paper, except for the tables and examples, j will be fixed and hence, when notationally convenient, we omit mention of it.

Prior to presenting the proof, we summarize well-known binomial identities used throughout the paper.

Proposition 1.2. *For any positive integer x,*

a) $\displaystyle\sum_{p=0}^{x} \binom{x}{p} = 2^x.$

b) $\displaystyle\sum_{p=0}^{x} (-1)^p \binom{x}{p} = 0.$

c) $\displaystyle\sum_{p=0}^{\frac{x-1}{2}} \binom{x}{p} = 2^{x-1}, \qquad \text{if } x \text{ is odd.}$

d) Furthermore, for any integer z, $1 \le z \le x-1$, and any integer $y, 0 \le y \le x$, we have

$$\binom{x}{y} = \sum_{p=0}^{x} \binom{z}{p}\binom{x-z}{y-p}. \tag{1.5}$$

Proof. Well known. For example, (d) follows by comparing coefficients in the expansions of both sides of the identity, $(1+V)^x = (1+V)^z(1+V)^{x-z}$. When $z=1$ we obtain the traditional Pascal Recursion. □

Proof. We now return to the proof of (1.4).

Equation (1.4) is equivalent to the j equations,

$$M_{k,*}G_n = j\begin{bmatrix} j(n+1)+l \\ k \end{bmatrix}_j - 2^{j(n+1)+l}, \qquad 0 \le k \le j-1. \tag{1.6}$$

Equation (1.6) implies that for each k, $M_{k.*}$ defines a linear homogeneous recursion with constant coefficients on the sequence $\{j\left[{j(n+1)+l \atop k}\right]_j - 2^{j(n+1)+l}\}_{n\ge1}$. Since the order-$j$ linear recursive sequences with constant coefficients form a vector space, to prove (1.6), it suffices to show that the recursion defined by $M_{k,*}$ holds for each summand in $j\left[{j(n+1)+l \atop k}\right]_j - 2^{j(n+1)+l}$. We deal separately with each summand. Since the second summand is straightforward, we deal with it first.

Second summand.

By (1.3), the rows of M are permutations of the binomial coefficients with $\binom{j}{0}$ and $\binom{j}{j}$ added together. Hence, by Proposition 1.2(a),

$$2^{j(n+1)+l} = \sum_{p=0}^{j} \binom{j}{p} 2^{jn+l}.$$

First summand. It suffices to prove

$$\begin{bmatrix} j(n+1)+l \\ k \end{bmatrix}_j = M_{k,*}\langle \begin{bmatrix} jn \\ q \end{bmatrix} : 0 \le q \le j-1\rangle, \qquad 0 \le k \le j-1. \tag{1.7}$$

By (1.3) and the identity $\binom{j}{x} = \binom{j}{j-x}$, we have

$$M_{k,*}(q) = \begin{cases} \binom{j}{k-q}, & \text{for } 0 \le q \le k-1, \\ \binom{j}{0} + \binom{j}{j}, & \text{for } q = k \\ \binom{j}{j-(q-k)}, & \text{for } k+1 \le q \le j-1. \end{cases} \tag{1.8}$$

Equation (1.1) shows that jump-sums are sums of binomial coefficients and hence they inherit the recursions satisfied by these binomial coefficients. Consequently, by (1.5),

$$\begin{aligned}
\begin{bmatrix} j(n+1)+l \\ k \end{bmatrix}_j &= \sum_{\substack{0 \le q \le j \\ p+q \equiv k\ (j)}} \binom{j}{p} \begin{bmatrix} jn+l \\ q \end{bmatrix}_j \\
&= \sum_{\substack{0 \le q \le k-1 \\ p+q \equiv k\ (j)}} \binom{j}{p} \begin{bmatrix} jn+l \\ q \end{bmatrix}_j \\
&+ \sum_{\substack{q=k \\ p+q \equiv k\ (j)}} \binom{j}{p} \begin{bmatrix} jn+l \\ q \end{bmatrix}_j \\
&+ \sum_{\substack{k+1 \le q \le j-1 \\ p+q \equiv k\ (j)}} \binom{j}{p} \begin{bmatrix} jn+l \\ q \end{bmatrix}_j .
\end{aligned} \tag{1.9}$$

Using (1.9), we can prove (1.7) by showing that for the 3 cases in (1.8) corresponding to the three summands on the right hand side of (1.9) we have that the sum of q with the bottom argument of the binomial coefficient is congruent to k modulo j. But for the top case we clearly have $k-q+q=k$, for the middle case we trivially have $k+0=k+j \equiv k\ (j)$, and for the bottom case we similarly have $j-(q-k)+q \equiv k\ (j)$. This completes the proof of (1.7) and hence of (1.6). $\square$

A similar proof, exploiting the fact that by (1.1) the jump-sum function inherits the recursions satisfied by the Pascal Triangle, yields the following proposition.

Proposition 1.3. *For any positive integers j, n and any non-negative integers $l, k, 0 \le l, k \le j-1$,*

$$\begin{bmatrix} jn+l+1 \\ k \end{bmatrix}_j = \begin{bmatrix} jn+l \\ k \end{bmatrix}_j + \begin{bmatrix} jn+l \\ k-1 \end{bmatrix}_j, \tag{1.10}$$

from which we derive

$$j\begin{bmatrix} jn+l+1 \\ k \end{bmatrix}_j - 2^{jn+l+1} = \left(j\begin{bmatrix} jn+l \\ k \end{bmatrix}_j - 2^{jn+l} \right) + \left(j\begin{bmatrix} jn+l \\ k-1 \end{bmatrix}_j - 2^{jn+l} \right). \tag{1.11}$$

From (1.7), *we directly have*

$$M\langle \begin{bmatrix} jn \\ k \end{bmatrix}_j : 0 \le k \le j-1 \rangle = \langle \begin{bmatrix} j(n+1) \\ k \end{bmatrix}_j : 0 \le k \le j-1 \rangle. \tag{1.12}$$

Tables 1 and 2 illustrate (1.10) and (1.11) respectively. Note that by (1.1), when applying the Pascal recursions of (1.10) and (1.11), k is interpreted modulo j so that -1 is interpreted as $j-1$.

Matrices are an established technique to derive recursions [8]. Equation (1.4) immediately gives us recursions satisfied by $\{G_n\}_{n \ge 1}$, since by letting $p = p_j$ be the characteristic polynomial of M_j, we have $p(M) = 0$, and therefore $p(M)G_n = 0$, for all $n \ge 1$. Consequently, $p(X)$ is the associated polynomial of a recursion satisfied by the vector sequence $\{G_n\}_{n \ge 1}$.

However, the degree of $p(X)$ is j while G in fact satisfies a recursion of order $\lfloor \frac{j-1}{2} \rfloor$. One approach to lowering the degree of p is to modify the Cayley-Hamilton polynomial by writing

$p'(X) = \prod(X - r_i)$, where the r_i are the *distinct* eigenvalues of p. This modified Cayley-Hamilton polynomial, under appropriate conditions (such as diagonalizability), also satisfies $p'(M) = 0$ [9] and hence $p'(M)G_n = 0$, for all $n \geq 1$. However, this too is not sufficient, since the degree of p' is greater (by 1 for odd j and by 2 for even j) than $\lfloor \frac{j-1}{2} \rfloor$. We must therefore extend the Cayley-Hamilton theory by studying polynomials, whose zeroes are a partial set of eigenvalues, evaluated at the underlying matrix.

This motivates the following outline of the rest of the paper. In Section 2, we present prerequisites summarizing important facts about circulants and values of polynomials whose roots are eigenvalues evaluated on the underlying matrices (Cayley-Hamilton theory). We also construct a modified Cayley-Hamilton polynomial, q. In Section 3, we show that although $q(M) \neq 0$, nevertheless, $q(M)G_n^{(j,l)} = 0$, for all j, n, l. Consequently, q is the associated polynomial of a recursion of order $\lfloor \frac{j-1}{2} \rfloor$. In Section 4, we derive exact formulas for some of the coefficients of $q(X)$.

2. Prerequisites

We need prerequisites on circulants, Vandermonde determinants, and Cayley-Hamilton.

The following proposition and definitions summarize major aspects of circulants [4].

Proposition 2.1. *Let $\zeta_j = e^{\frac{2\pi i}{j}}$, be a $j-th$ root of unity. Then the eigenvalues of any $j \times j$ circulant matrix are given by the following.*

$$\lambda_k = \sum_{p=0}^{j} \binom{j}{k} \zeta^{pk}, \qquad 0 \leq k \leq j-1. \tag{2.1}$$

Define the Vandermonde matrix V_j by

$$\sqrt{j} V_{i,k} = \zeta^{ik}, \qquad 0 \leq i, k \leq j-1. \tag{2.2}$$

Then $V^{-1} = \overline{V}$ and

$$M_j = V_j D_j V_j^{-1}, \tag{2.3}$$

with $D = D_j$ the diagonal matrix of eigenvalues of M_j, with

$$D_j(i,i) = \lambda_i, \qquad 0 \leq i \leq j-1. \tag{2.4}$$

.

Corollary 2.2. *The eigenvalues of M are given by* (2.1).

Proof. Proposition 2.1 applies to any $j \times j$ circulant and hence by (1.3) applies to M. □

The following proposition summarizes some basic facts about the eigenvalues, λ_k.

Proposition 2.3.
a) $\lambda_k = \sum_{i=0}^{j} \binom{j}{i} \zeta_j^{ki} = (1 + \zeta_j^k)^j$.
b) $\lambda_0 = 2^j$.
c) $\lambda_{\frac{j}{2}} = 0$, *if j is even.*
d) $\lambda_k = \lambda_{j-k}, k \neq 0$.
e) λ_0 has multiplicity 1; when j is even, $\lambda_{\frac{j}{2}}$ has multiplicity 1; all other roots have multiplicity 2.

Proof. (a) follows from (2.1) and the binomial expansion as shown. (b)-(d) follow from (a) and Proposition 1.2. (e) follows from (a). For example, $(1+\zeta^k)^j = 2^j$ requires $\zeta_k = 1$ which requires $k = 0$; hence, λ_0 has multiplicity 1. □

We now present propositions about polynomials evaluated at matrices.

Proposition 2.4. *Let $B = HEH^{-1}$ be a matrix equation about $m \times m$ matrices. Let $r(X)$ be any polynomial. Then $r(B) = Hr(E)H^{-1}$.*

Proof. The proposition is clearly true for polynomials of the form $r(x) = X^t$ and hence extends to arbitrary polynomials by scalar multiplication and addition. □

Define the corner matrix $C = C(x)$, by

$$C_x(i,j) = \begin{cases} x, & \text{if } (i,j) = (0,0), \\ 0, & \text{if } (i,j) \neq (0,0). \end{cases} \tag{2.5}$$

The corner matrices are useful because of the following proposition.

Proposition 2.5. *With $D = D_j$ defined by (2.4) and λ_k defined by (2.1), define a polynomial $q = q_j(x)$ by*

$$q(X) = \begin{cases} \displaystyle\prod_{k=1}^{\frac{j-1}{2}} (X - \lambda_k), & \text{if } j \text{ is odd}, \\ \displaystyle\prod_{k=1}^{\frac{j}{2}-1} (X - \lambda_k), & \text{if } j \text{ is even}. \end{cases} \tag{2.6}$$

Then

$$q(D) = C(q(\lambda_0)). \tag{2.7}$$

Comment 2.6. *The zeroes of $q(X)$ are the eigenvalues of M, without multiplicity, except for $\lambda_0 = 2^j$ and except for $\lambda_{\frac{j}{2}} = 0$, when j is even. We prove in Section 3 that even though $q(M) \neq 0$, $q(M)G_n = 0, n \geq 1$. Consequently, q_j is the associated polynomial of a recursion of order $\lfloor \frac{j-1}{2} \rfloor$ satisfied by the vector sequence $\{G_n\}_{n\geq 1}$. We thus see that $q(X)$ is the desired modification of the Cayley-Hamilton polynomial. Therefore, prior to the proof of Proposition 2.4, it might be worthwhile to see some examples.*

Example 2.7. *Let $j = 3$. Then by (2.1), the three eigenvalues of $M_3 = \mathbf{Circ}(2,3,3)$ are $\lambda_0 = 2^3 = 8$, $\lambda_1 = \lambda_2 = 2 + 3\omega + 3\omega^2$, with ω a primitive cube root of unity. In this case $q_3(X) = (X - \lambda_1)$. But $1+\omega+\omega^2 = 0$ implying that $\lambda_1 = -1$, and consequently $q_3(x) = X+1$, which is the associated polynomial of the recursion $G_n + G_{n-1} = 0$, which as we saw in Section 1, is satisfied by the vector sequence $\{G_n^{(3,0)}\}_{n\geq 1}$.*

Example 2.8. *Let $j = 4$. Then by (2.1), the 4 eigenvalues of $M_4 = \mathbf{Circ}(2,4,6,4)$, are $\lambda_0 = 16$, $\lambda_1 = \lambda_3 = 1 + 4i + 6i^2 + 4i^3 + i^4 = -4$, and $\lambda_2 = 1 + 4i^2 + 6i^4 + 4i^6 + i^8 = 0$. In this case, $q_4(X) = (X - \lambda_1) = X + 4$. One can check that $G_n + 4G_{n-1} = 0$.*

One can write down the coefficients of $q_i, i = 3, 4, \ldots$, with one polynomial per row. This gives rise to the jump sum recursion triangle [11], displayed in Table 5. The closed functional forms $2^{j-1} - j$ for the second column and $(\frac{j}{2})^{\frac{j}{2}}$ for right-most diagonal on rows where j is even, will be proven in Section 4.

j	Coefficients of $q_j(X)$
3	1, 1
4	1, 4
5	1, 11, -1
6	1, 26, -27
7	1, 57, -289, -1
8	1, 120, -2160, -256
9	1, 247, -13359, -13604, 1
10	1, 502, -73749, -383750, 3125

TABLE 5. Coefficients of $q_j(X)$, (2.6), in descending exponent order. The last coefficient has degree 0. For example, $q_5(X) = X^2 + 11X - 1$. $q(X)$ is the associated polynomial of a recursion on the vectors $\{G_n\}_{n\geq 1}$, (1.2). For example, if $j = 5$, $G_n + 11G_{n-1} - G_{n-2} = 0$, $n \geq 1$.

Proof. We return to the proof of Proposition 2.4. We may think of D as arranged in blocks of λ_i. By Proposition 2.3(e), λ_0 has multiplicity 1 so the upper left block has dimensions 1×1. Consider the effect of the factor $X - \lambda_i$ on M. (i) The upper left cell has $\lambda_0 - \lambda_i$, (ii) the block with λ_i down the diagonal has all zeroes, and (iii) other blocks have $\lambda_k - \lambda_i$ down the diagonal. Upon multiplication, we have zeroes in all blocks except the leftmost cell which has $(\lambda_0 - \lambda_1)(\lambda_0 - \lambda_2)\ldots = q(\lambda_0)$ as was to be shown. □

We need one more concept. Besides $I = I_j$ which is the $j \times j$ identity matrix we need a matrix $J = J_j$ defined as follows.

$$J_{x,y} = 1, \qquad 0 \leq x, y \leq j - 1. \tag{2.8}$$

We have the following elementary results.

Proposition 2.9.
a) $J^2 = jJ$.
b) $JM = MJ = 2^j J$.
c) For any positive integer n, $JM^n = M^n J = 2^{jn} J$
d) With V,C *and* q *defined by* (2.2),(2.5) *and* (2.6) *and for any complex* z_0, *we have* $q(VC(z_0)V^{-1}) = \frac{1}{j} q(z_0)J$.

Proof. (a) and (b) are clear. For example, to prove (b), all the rows of J are ones and hence the entries of JM are dot products of a vector of ones with the binomial coefficients in some permutation and therefore equal to 2^j by Proposition 1.2(a). (c) Follows from (b) by a routine induction. (d) follows from the fact that V and $\overline{V} = V^{-1}$ have a left column and top row of all ones. The $\frac{1}{j}$ comes from the normalization factor in (2.2). □

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Proposition 2.10. *With* $M = M_j$ *defined by* (1.3) *and* $q = q_j$ *defined by* (2.6), *we have*

$$q(M) = \frac{1}{j} q(2^j)J. \tag{2.9}$$

Proof.

$$\begin{aligned} q(M) =& q(VDV^{-1}), && \text{by (2.3)},\\ =& Vq(D)V^{-1}, && \text{by Proposition 2.4},\\ =& VC(q(\lambda_0))V^{-1}, && \text{by Proposition 2.5},\\ =& VC(q(2^j))V^{-1}, && \text{by Proposition 2.3(b)},\\ =& \frac{1}{j}q(2^j)J, && \text{by Proposition 2.9(d)}. \end{aligned}$$

□

3. The Main Theorem

Theorem 3.1. *For any integers $n \geq 1$, $j \geq 3$, and $0 \leq l \leq j-1$, and with q, M, and G_n defined by* (2.6)*,* (1.3)*, and* (1.2) *respectively, we have*

$$q_j(M_j)G_n^{(j,l)} = 0, \qquad n \geq 1. \tag{3.1}$$

Corollary 3.2. *For fixed j, l, $q_j(X)$ is the associated polynomial to a recursion satisfied by the $\{G_n^{(j,l)}\}_{n\geq 1}$.*

Proof. We first prove (3.1) assuming $l = 0$.

By (1.2), (3.1) is equivalent to

$$q(M)\langle j \begin{bmatrix} jn \\ k \end{bmatrix}_j : 0 \leq k \leq j-1\rangle = q(M)\langle 2^{jn} : 0 \leq k \leq j-1\rangle. \tag{3.2}$$

But, by (2.8),

$$q(M)\langle 2^{jn} : 0 \leq k \leq j-1\rangle = q(M)2^{jn}J_{*,0}, \tag{3.3}$$

and similarly by (1.2) and (1.8) evaluated at $k = 0$,

$$G_1 = \langle \binom{j}{0} + \binom{j}{j}, \binom{j}{1}, \binom{j}{2}, \ldots, \binom{j}{j-1}\rangle = M_{0,*}. \tag{3.4}$$

Hence, by (1.12),

$$q(M)\langle j \begin{bmatrix} jn \\ k \end{bmatrix}_j : 0 \leq k \leq j-1\rangle = q(M)jM^{n-1}M_{0,*}. \tag{3.5}$$

In proving (3.2), a crucial step is replacement of the vectors in (3.3) and (3.5) by matrices. In other words, by (3.2)-(3.5), to prove (3.1) it *suffices* to prove

$$q(M)jM^{n-1}M = q(M)2^{jn}J. \tag{3.6}$$

.

We prove (3.6) by showing the left and right sides equal. By (2.9) and Proposition 2.9(c), we have

$$q(M)jM^n = jq(M)M^n = j\frac{1}{j}q(2^j)JM^n = q(2^j)2^{jn}J.$$

Similarly, by (2.9) and Proposition 2.9(a), we have

$$q(M)2^{jn}J = 2^{jn}\frac{1}{j}q(2^j)JJ = 2^{jn}\frac{1}{j}q(2^j)jJ = q(2^j)2^{jn}J.$$

This completes the proof of (3.1) when $l = 0$.

To prove (3.1) for $l > 1$, we use an inductive argument. We assume (3.1) is proven for the case l and proceed to prove it for the case $l+1$. The base case occurs when $l = 0$. But by (1.11), if a recursion holds for l then it holds for $l+1$.

This completes the proof of the Main Theorem. □

4. Coefficient Results

Certain patterns emerge for the second and last coefficient in the jump sum triangle displayed in Table 5. We formally state them as a corollary to the Main Theorem.

Corollary 4.1. *With q defined by (2.6), let $m = \lfloor \frac{j-1}{2} \rfloor$. Further, define c_i by*

$$q(X) = X^m + c_1 X^{m-1} + c_2 X^{m-2} + \ldots + c_m. \tag{4.1}$$

Then

$$c_1 = 2^{j-1} - j. \tag{4.2}$$

Furthermore, if j is even, then

$$|c_m| = \left(\frac{j}{2}\right)^{\frac{j}{2}}. \tag{4.3}$$

.

Proof. Technically, by (2.6), to prove (4.2), we have to consider j even and odd separately. To prove (4.2), we assume j odd, the proof for the even case being similar and hence omitted. To prove (4.3), we assume j even. By (2.6), (4.1), (2.1) and Proposition 2.3(a), we have

$$-c_1 = \sum_{k=1}^{\frac{j-1}{2}} \lambda_k = \sum_{k=1}^{\frac{j-1}{2}} \sum_{p=0}^{j} \binom{j}{p} \zeta^{pk}; \qquad c_m = \prod_{p=1}^{\frac{j}{2}-1} \lambda_p = \prod_{p=1}^{\frac{j}{2}-1} (1+\zeta_j^p)^j. \tag{4.4}$$

Proof of (4.2)**.** In (4.4) we may interchange the order of summation and carve out the 0 and j term separately.

$$\begin{aligned} -c_1 &= \sum_{p=0}^{j} \binom{j}{p} \sum_{k=1}^{\frac{j-1}{2}} \zeta^{pk} \\ &= \sum_{p=0} \binom{j}{p} \sum_{k=1}^{\frac{j-1}{2}} \zeta^{pk} + \sum_{p=j} \binom{j}{p} \sum_{k=1}^{\frac{j-1}{2}} \zeta^{pk} + \sum_{p=1}^{j-1} \binom{j}{p} \sum_{k=1}^{\frac{j-1}{2}} \zeta^{pk} \\ &= j - 1 + \sum_{p=1}^{j-1} \binom{j}{p} \sum_{k=1}^{\frac{j-1}{2}} \zeta^{pk}. \end{aligned} \tag{4.5}$$

For the last summand in (4.5), since $\binom{j}{k} = \binom{j}{j-k}$, we have

$$\sum_{p=1}^{j-1} \binom{j}{p} \sum_{k=1}^{\frac{j-1}{2}} \zeta^{pk} = \sum_{p=1}^{\frac{j-1}{2}} \binom{j}{p} \sum_{k=1}^{\frac{j-1}{2}} \left(\zeta^{pk} + \zeta^{-pk}\right). \tag{4.6}$$

p is fixed in the inner summand. Since we are looking at exponents of j-th roots of unity we can evaluate these exponents modulo j. Let g equal the greatest common divisor of p and j. Then, for each fixed p, and evaluating modulo j, we have

$$\{0\}\bigcup\{kp : 1 \leq k \leq \frac{j-1}{2}\}\bigcup\{-kp : 1 \leq k \leq \frac{j-1}{2}\} = \{0, g, 2g, 3g, \ldots, (\frac{j}{g} - 1)g\}.$$

Hence,

$$\zeta^0 + \sum_{k=1}^{\frac{j-1}{2}} \left(\zeta^{pk} + \zeta^{-pk}\right) = 0. \tag{4.7}$$

Applying (4.7) to (4.6) and using Proposition 1.2(c), we have

$$\sum_{p=1}^{j-1} \binom{j}{p} \sum_{k=1}^{\frac{j-1}{2}} \zeta^{pk} = \sum_{p=1}^{\frac{j-1}{2}} \binom{j}{p}(-1) = -(2^{j-1} - 1). \tag{4.8}$$

Equation (4.2) now follows from (4.1), (4.5) and (4.8).

Proof of (4.3)**.** Since j is assumed even, let

$$j = 2n. \tag{4.9}$$

By (4.4), (4.9) and Proposition 2.3(d),

$$c_m = \prod_{p=1}^{n-1} \lambda_p = \prod_{p=1}^{n-1} \lambda_{2n-p} = \prod_{p=1}^{n-1} (1 + \zeta_{2n}^{p+n})^j = \prod_{p=1}^{n-1} (1 - \zeta_{2n}^{p})^j. \tag{4.10}$$

Combining (4.4),(4.10) with Proposition 2.3(a) and using the identity $(1 - \zeta_{2n}^p)(1 + \zeta_{2n}^p) = (1 - \zeta_n^p)$, we have

$$c_m^2 = \prod_{p=1}^{n-1} \lambda_p \prod_{p=1}^{n-1} \lambda_{2n-p} = \prod_{p=1}^{n-1} (1 - \zeta_{2n}^p)^j \prod_{p=1}^{n-1} (1 + \zeta_{2n}^p)^j = \prod_{p=1}^{n-1} (1 - \zeta_n^p)^j. \tag{4.11}$$

To evaluate the last product we use the formula for geometric series and the fundamental theorem of algebra, to obtain

$$1 + X + \ldots X^{n-1} = \frac{X^n - 1}{X - 1} = (X - \zeta_n)(X - \zeta_n^2) \ldots (X - \zeta_n^{n-1}).$$

Letting $X = 1$ in the last equation, yields

$$n = \prod_{p=1}^{n-1} (1 - \zeta_n^p). \tag{4.12}$$

By (4.11), (4.9) and (4.12) we have $c_m^2 = n^{2n} = \left(\frac{j}{2}\right)^j$, proving (4.1). □

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