PROBLEM PROPOSALS

COMPILED BY CLARK KIMBERLING

These fourteen problems were posed by participants of the Sixteenth International Conference on Fibonacci Numbers and Their Applications, Rochester Institute of Technology, Rochester, New York, July 24, 2014. A few solutions and partial solutions, received during August-December, are included.

Problem 1, posed by Marjorie Johnson

Prove or disprove that the only Pythagorean triples containing exactly two Fibonacci numbers are $3, 4, 5$ and $5, 12, 13$.

Problem 2, posed by Heiko Harborth and Jens-P. Bode

Two players A and B choose alternatingly an integer. Does there exist a strategy for A to choose integers n, $n+2$, $n+3$, and $n+5$ for some n, or, equivalently, does there exist a strategy for B to prevent A from this objective?

Problem 3, posed by Clark Kimberling

Observe that

$$\begin{aligned} &1/6 + 1/7 + 1/8 \\ &< 1/9 + \cdots + 1/13 \\ &< 1/14 + \cdots + 1/21 \\ &< 1/22 + \cdots + 1/34 \end{aligned}$$

Let $H(n) = 1/1 + 1/2 + \cdots + 1/n$, so that the observation can be written using Fibonacci numbers as

$$\begin{aligned} &H(8) - H(5) \\ &< H(13) - H(8) \\ &< H(21) - H(13) \\ &< H(34) - H(21) \end{aligned}$$

More generally, if $x \leq y$, let

$$\begin{aligned} a(1) &= \text{least } k \text{ such that } H(y) - H(x) < H(k) - H(y); \\ a(2) &= \text{least } k \text{ such that } H(a(1)) - H(y) < H(k) - H(a(1)); \\ a(n) &= \text{least } k \text{ such that } H(a(n-1)) - H(a(n-2)) \\ &\qquad < H(k) - H(a(n-1)), \end{aligned}$$

for $n \geq 3$. Prove that if $(x, y) = (5, 8)$, then $a(n) = F(n+6)$, and determine all (x, y) for which $(a(n))$ is linearly recurrent.

Problem 4, posed by Peter Anderson

Let $u_{n+1} = u_n + u_{n-k}$, for $n \geq k$, where $u_i = 0$ for $i = 0, 1, \ldots, k-1$ and $u_k = 1$. Let α be the largest real root of the companion polynomial. For $k = 3$, show how to obtain the Bergman representation of every positive integer. For $k > 4$, show that there is no finite Bergman representation of 2.

Problem 5.1, posed by Dale Gerdemann (problem 5, version 1, as proposed in Rochester)

In "Bergman-Fibonacci" representation,

$$\begin{aligned}1 &= 1.0 = 1 + 0\\ 2 &= 1.01 = 1 + 1\\ 3 &= 10.01 = 2 + 1\\ 4 &= 101.01 = 3 + 1\\ 5 &= 1000.1001 = 3 + 2.\end{aligned}$$

What is the ratio of the values of the positive digits to the value of the negative digits? Does it approach a limit?

Empirical solution by Margaret P. Kimberling, Lynda J. Martin, and Peter J. C. Moses. "Bergman-Fibonacci" representations use base $\varphi = (1 + \sqrt{5})/2$ with φ^n replaced by F_{n+1}, so that the five examples are interpreted as

$$\begin{aligned}1 &= F_1 + F_0 = 1 + 0\\ 2 &= F_2 + F_{-1} = 1 + 1\\ 3 &= F_3 + F_{-1} = 2 + 1\\ 4 &= F_3 + F_1 + F_{-1} = 3 + 1\\ 5 &= F_4 + (F_0 + F_{-3}) = 3 + 2.\end{aligned}$$

Thus each n has a representation of the form $x.y = u + v$, where u and v are the positive part and negative part, respectively. We claim that the ratios are given by

$$u/v = u(n)/v(n) = (1 + k)/(n - k - 1),$$

where $k = \lfloor (n-1)/(3-\varphi) \rfloor$, and that $\lim_{n\to\infty} u(n)/v(n) = \varphi + 1$.

The claim is based on the following Mathematica code, which finds the Bergman-Fibonacci representation of n, using the first 1000 base 10 digits of φ:

```
phiBase[n_] := Last[#] - Flatten[Position[First[#], 1]] &
[RealDigits[n, GoldenRatio, 1000]];
```

To see the representation for an example, say $n = 12$, add this line of code:

```
test = 12; SplitBy[phiBase[test] + 1, # > 0 &]
```

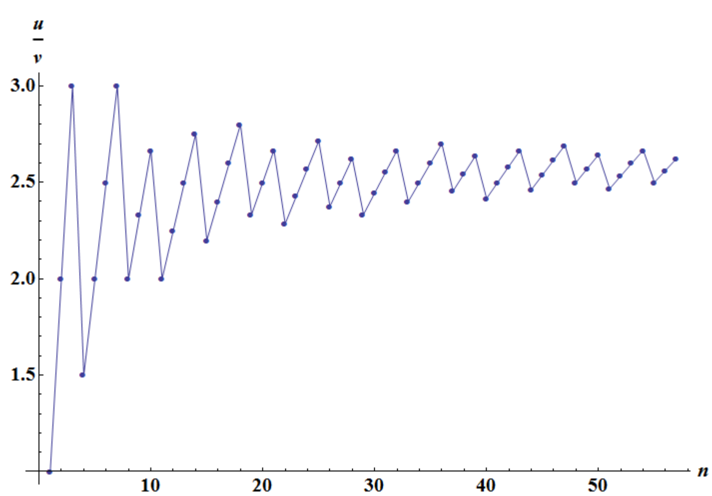
which shows $\{\{6\}, \{0, -2, -1\}\}$, i.e.,

$$F_6 + (F_0 + F_{-2} + F_{-3}) = 8 + [0 + (-1) + 5] = 12.$$

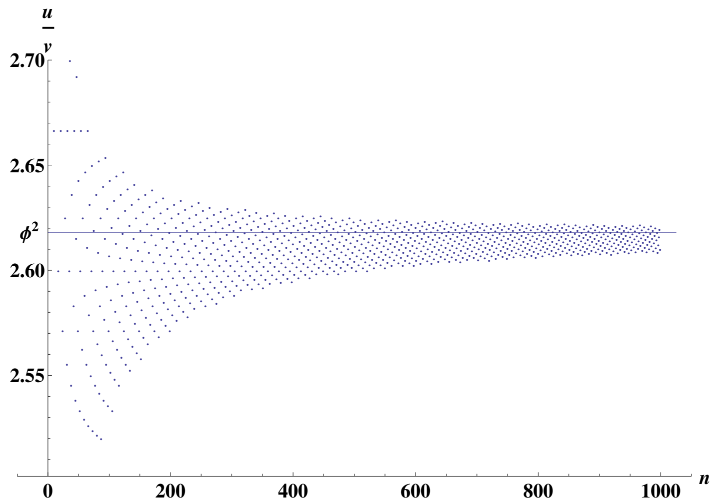
In this example, $u = 8$ and $v = 0 + (-1) + 5 = 4$. The code can be extended to obtain

2	10.01	1/1
3	100.01	2/1
4	101.01	3/1
5	1000.1001	3/2
6	1010.0001	4/2
7	10000.0001	5/2
8	10001.0001	6/2
9	10010.0101	6/3
10	10100.0101	7/3
11	10101.0101	8/3
12	100000.101001	8/4
13	100010.001001	9/4
14	100100.001001	10/4
15	100101.001001	11/4
16	101000.100001	11/5

The denominators in column 3 form the sequence $(1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 4, \ldots)$, of which the difference sequence is $(0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, \ldots)$, which appears to be the sequence indexed in OEIS [2] as A221150, authored by Neil Sloane in 2013. Information given at A221150 enables use to find $v(n) = n + \lfloor (n-1)/(\varphi - 3) \rfloor$. The difference sequence for the numerators appears to be the binary complement of A221150, $(1, 1, 0, 1, 1, 1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 0, \ldots)$, leading to $u(n) = 1 + \lfloor (n-1)/(3 - 3\varphi) \rfloor$. The fractions u/v for $2 \le n \le 60$ and $2 \le n \le 1000$ are depicted here:



u/v for $2 \le n \le 60$



u/v for $2 \le n \le 1000$

As a somewhat randomly selected example take $n = 46792386$. Then **SplitBy[phiBase[test] + 1, # > 0 &] gives**

$$\{\{37, 35, 28, 26, 23, 19, 17, 13, 11, 9, 6, 4\},$$
$$\{0, -5, -7, -9, -11, -15, -17, -22, -27, -33, -35\}\},$$

which represents

$$\begin{aligned} u &= F_{37} + F_{35} + F_{28} + F_{26} + F_{23} + F_{19} + F_{17} + F_{13} + F_{11} \\ &\quad + F_9 + F_6 + F_4 \\ &= 33859288; \\ v &= F_0 + F_{-5} + F_{-7} + F_{-9} + F_{-11} + F_{-15} + F_{-17} + F_{-22} + F_{-27} \\ &\quad + F_{-33} + F_{-35} \\ &= 12933098. \end{aligned}$$

It can now be checked that u/v in this case agrees with the asserted formula. Finally, it is easy to check that if $u(n)$ and $v(n)$ are as asserted, then $\lim_{n\to\infty} u(n)/v(n) = \varphi + 1$.

References.

[1] Bergman, G. *A number system with an irrational base,* Mathematics Magazine 31 (1957-58) 98-110.

[2] Online Encyclopedia of Integer Sequences, **https://oeis.org/**

Partial solution by Dale Gerdemann. This is a partial proof of the following statement: In Bergman-Fibonacci representation (golden ratio base with each φ^n replaced by f_n, where $f_0 = 1$, $f_1 = 1$, $f_n = f_{n-1} + f_{n-2}$, $f_n = f_{n+2} - f_{n-1}$), the ratio of the positively indexed Fibonacci numbers to the negatively indexed Fibonacci numbers converges to $\varphi + 1$. I limit myself here to proving the weaker statement that if this sequence converges, then it converges to $\varphi + 1$. My strategy is to find a convergent subsequence consisting of the simplest Bergman-Fibonacci representations and then to employ this basic fact about limits: *Every subsequence of a convergent sequence converges, and its limit is the limit of the original sequence.*

The simplest Bergman-Fibonacci representations are for the odd-indexed Lucas numbers, and the second simplest are for the even indexed Lucas numbers, where the Lucas numbers are indexed here starting with $L_0 = -1$ and $L_1 = 2$:

$$\begin{aligned} L_{2n-1} &= f_{2n-2} + f_{-2n+2} \\ L_{2n} &= f_{2n-2} + f_{2n-4} + \cdots + f_0 + \cdots + f_{-2n+4} + f_{-2n+2}. \end{aligned}$$

For an inductive proof, note that these two statements are true for the following base case: $L_4 = f_2 + f_0 + f_{-2}$. For the inductive steps, note that

$$\begin{aligned} L_{2n+1} &= L_{2n-1} + L_{2n} \\ &= 2f_{2n-2} + f_{2n-4} + \cdots + f_0 + \cdots + f_{-2n+4} + 2f_{-2n+2} \\ &= f_{2n} + f_{-2n} \\ L_{2n+2} &= L_{2n+1} + L_{2n} \\ &= f_{2n} + f_{2n-2} + \cdots + f_0 + \cdots + f_{-2n+2} + f_{-2n}. \end{aligned}$$

Now consider the positive-indexed to negative-indexed ratio for the subsequence of odd-indexed Lucas numbers:

$$\cdots \frac{f_{2n-2}}{f_{-2n+2}}, \frac{f_{2n}}{f_{-2n}}, \frac{f_{2n+2}}{f_{-2n-2}}, \ \cdots .$$

Since the denominators are even-indexed, the negative indexing can be eliminated:

$$\cdots \frac{f_{2n-2}}{f_{2n-4}}, \frac{f_{2n}}{f_{2n-2}}, \frac{f_{2n+2}}{f_{2n}}, \ \cdots .$$

Now, as is well known, the ratio of adjacent Fibonacci numbers converges to φ, so that the ratio of these two-apart Fibonacci numbers converges to $\varphi^2 = \varphi + 1$.

Problem 5.2, posed by Dale Gerdemann (problem 5, version 2)

Golden ratio base differs from more familiar integer bases in that it uses both positive and negative powers of the base to represent an integer. For example, the number $m = 100$ is represented as the sum

$$\varphi^9 + \varphi^6 + \varphi^3 + \varphi + \varphi^{-4} + \varphi^{-7} + \varphi^{-10},$$

where $\varphi = (1+\sqrt{5})/2$. Here the contribution of the positive powers is much greater than the contribution of the negative powers. Note what happens, however, when the powers of φ are replaced by corresponding Fibonacci numbers (using the combinatorial definition: $f_0 = 1$, $f_1 = 1$, $f_n = f_{n-1} + f_{n-2}$, $f_n = f_{n+2} - f_{n-1}$):

$$\begin{aligned} &f_9 + f_6 + f_3 + f_1 + f_{-4} + f_{-7} + f_{-10} \\ &= 55 + 13 + 3 + 1 + 2 - 8 + 34 \\ &= 100 \end{aligned}$$

This replacement does not change the sum, which remains 100. However, the negatively indexed Fibonacci numbers play a larger role than the corresponding negative powers in golden ratio base. Here the positively indexed Fibonacci numbers sum to 72, the negative ones sum to 28, and the ratio $72/28 = 2.571...$. Prove that as m increases, this ratio approaches $\varphi + 1$.

Problem 6, posed by Curtis Cooper

The following two statements are true: If $g^5 = 2$, then

$$\frac{\sqrt[3]{5g^2+1}+\sqrt[3]{35g^2+g-43}}{\sqrt[3]{5g^2+1}-\sqrt[3]{35g^2+g-43}}=\frac{2+g-g^2}{-g+g^2},$$

and if $g^7 = 2$, then

$$\frac{\sqrt[5]{15g^3+11g^2+15g+12}+\sqrt[5]{-270g^4-259g^3+346g^2+315g+14}}{\sqrt[5]{15g^3+11g^2+15g+12}-\sqrt[5]{-270g^4-259g^3+346g^2+315g+14}}$$
$$=\frac{2+g-g^2}{-g+g^2}.$$

Find similar true statements for $g^k = 2$ where $k \geq 9$ is an odd integer.

Solutions by Sam Northshield for $k = 7, 9, 11,$ **and** 13. For $k = 7$, we present a solution distinct from the one stated just above (g^4 does not appear in our new solution).

$$\frac{\sqrt[5]{A}+\sqrt[5]{B}}{\sqrt[5]{A}-\sqrt[5]{B}}=\frac{2+g-g^2}{-g+g^2}\text{ if }g^7=2,$$

$$A = -15g^3 - 6239g^2 + 255g - 6438,$$
$$B = 112561g^3 + 20246g^2 - 160155g - 6836.$$

$$\frac{\sqrt[7]{A}+\sqrt[7]{B}}{\sqrt[7]{A}-\sqrt[7]{B}}=\frac{2+g-g^2}{-g+g^2}\text{ if }g^9=2,$$

$$A = 8980553g^4 + 7941290g^3 + 15149890g^2 + 6386905g + 11823140,$$
$$B = -45991056g^4 - 420491442g^3 - 440508591g^2 + 579500187g + 511466434.$$

$$\frac{\sqrt[9]{A}+\sqrt[9]{B}}{\sqrt[9]{A}-\sqrt[9]{B}}=\frac{2+g-g^2}{-g+g^2}\text{ if }g^{11}=2,$$

$$A = a_5g^5 + a_4g^4 + a_3g^3 + a_2g^2 + a_1g + a_0,$$
$$B = b_5g^5 + b_4g^4 + b_3g^3 + b_2g^2 + b_1g + b_0,$$
$$b_5 = -2832370277, b_4 = 2254685169, b_3 = 4298350067,$$
$$b_2 = -4610451384, b_1 = -2556248098, b_0 = 3738894258,$$
$$a_5 = 1050574, a_4 = -915414, a_3 = 9829317,$$
$$a_2 = -12489450, a_1 = 8175912, a_0 = -8267688.$$

$$\frac{\sqrt[11]{A}+\sqrt[11]{B}}{\sqrt[11]{A}-\sqrt[11]{B}}=\frac{2+g-g^2}{-g+g^2}\text{ if }g^{13}=2,$$

$$
\begin{aligned}
&A = a_6g^6 + a_5g^5 + a_4g^4 + a_3g^3 + a_2g^2 + a_1g + a_0,\\
&B = b_6g^6 + b_5g^5 + b_4g^4 + b_3g^3 + b_2g^2 + b_1g + b_0,\\
&b_6 = 18362391990345, b_5 = 10955091993365, b_4 = -54313592877440,\\
&b_3 = -15431135576532, b_2 = 68772473586419, b_1 = 5062921298005,\\
&b_0 = -36107722357990, a_6 = -22949055914, a_5 = 10769387302,\\
&a_4 = -30534819159, a_3 = 46896418382, a_2 = -29067883130,\\
&a_1 = 33833389975, a_0 = -6861047820.
\end{aligned}
$$

$$\frac{\sqrt[13]{A} + \sqrt[13]{B}}{\sqrt[13]{A} - \sqrt[13]{B}} = \frac{2 + g - g^2}{-g + g^2} \text{ if } g^{15} = 2,$$

$$
\begin{aligned}
&A = a_7g^7 + a_6g^6 + a_5g^5 + a_4g^4 + a_3g^3 + a_2g^2 + a_1g + a_0,\\
&B = b_7g^7 + b_6g^6 + b_5g^5 + b_4g^4 + b_3g^3 + b_2g^2 + b_1g + b_0,\\
&b_7 = 124297024336997477790866, b_6 = 431896224562463934 14224,\\
&b_5 = -258642712235742814743726, b_4 = -136428165027671534593750,\\
&b_3 = 213681762408969527031250, b_2 = 199976876120553414562602,\\
&b_1 = -707744166102550879517 47, b_0 = -129089005248346771092897,\\
&a_7 = 5916530127295603752 5, a_6 = 40996615699845889982,\\
&a_5 = 15296900530162287448 9, a_4 = 53665185894144140125,\\
&a_3 = 14845381020449621037 5, a_2 = 33066388703204638925,\\
&a_1 = 63819073735217372000, a_0 = 23375208041868016 75.
\end{aligned}
$$

Method: I used Maple, which handles large integers easily.

1) Find remainder of

$$(1 + g - g^2)^{2n-1}(a_ng^n + ... + a_0)$$

upon division by $g^{2n+1} - 2$ (where g and all the a_i's are indeterminate). The result is a polynomial, in g, of degree $2n$ with each coefficient b_j a linear combination of the a_i's.

2) Solving $b_{2n} = ... = b_{n+1} = 0$ and $b_n = k$ gives, for the right choice of k, relatively prime integers $a_0, ..., a_n$.

3) Letting $A(g) = \sum a_ig^i$, and letting $B(g) = \sum b_ig^i$ be the remainder of $(1+g-g^2)^{2n-1}A(g)$ upon division by $g^{2n+1} - 2$, implies

$$(1 + g - g^2)^{2n-1}A(g) = (g^{2n+1} - 2)P(g) + B(g)$$

for some polynomial $P(g)$. If $g^{2n+1} = 2$, then

$$B(g)/A(g) = (1 + g - g^2)^{2n-1}$$

or, equivalently,

$$\frac{\sqrt[2n-1]{A(g)} + \sqrt[2n-1]{B(g)}}{\sqrt[2n-1]{A(g)} - \sqrt[2n-1]{B(g)}} = \frac{2 + g - g^2}{-g + g^2}.$$

Problem 7, posed by Sam Northshield

Let

$$f(n+1) = \sum_{k=0}^{n} \sigma^{S_F(k)} \overline{\sigma}^{S_F(n-k)},$$

where $\sigma = (1+i\sqrt{3})/2$, and $S_F(k)$ is the number of terms in the Zeckendorf representation of k. The sequence f begins:

$$1, 1, 2, 3, 2, 4, 3, 3, 6, 4, 6, 6, 4, 8, 6, 7, \ldots$$

and is integer-valued. Define $\sigma(n) = \lfloor n\varphi + 1/\varphi \rfloor$, where $\varphi = (1+\sqrt{5})/2$ and $\tau(n) = \lfloor n\varphi^2 + \varphi \rfloor$, so that these sequences are a complementary pair. Prove or disprove the following chain of inequalities:

$$f(\tau(n)) \geq f(\lfloor n\varphi^2 \rfloor) \geq f(\lfloor n\varphi \rfloor) \geq f(\sigma(n)) \geq f(n) \geq 0.$$

Also, what does the sequence f count?

Problem 8, posed by Larry Ericksen

Let $p_i = \sum_{j=0}^{J_i} c_j 10^j$ be the decimal representation of the ith prime, and let $r_i = \sum_{j=0}^{J_i} 10^{J_i - j} c_j$ be the number obtained by reversing the digits. For what primes p_i is $r_i + p_i$ a square and $r_i - p_i$ a cube? Example: for $p_i = 47$ and $r_i = 74$, we have $r_i + p_i = 11^2$ and $r_i - p_i = 3^3$.

Problem 9, posed by Patrick Dynes

It is known that the sequence of Fibonacci numbers modulo q, where $q \in \mathbb{Z}^+$, repeats with period $\pi(q)$, known as the Pisano period. Given integers $0 \leq r < q$ and n, let $S(q,r,n) = \{F_i : i \leq n \text{ and } F_i \equiv r \pmod{q}\}$. How well can we approximate $|S(q,r,n)|$? Is it possible to develop an asymptotic formula for $|S(q,r,n)|$ that becomes more precise as q and n grow arbitrarily large?

Problem 10, posed by Russell Hendel

Let $\{a_{n,i}\}_{n\geq 0}$, $1 \leq i \leq m$, be a collection of m linear homogeneous recursive nondecreasing sequences with constant coefficients. Define the *merged sequence* as the sequence formed by arranging in nondecreasing order the set-theoretic union of these sequences. Define the weight, w, of a sequence $\{G_n\}_{n\geq 0}$ satisfying $\sum_{i=0}^{p} b_i G_{n-i} = 0$ by $w_G = \sum_{i=0}^{p} |b_i|$.

Problem: Under what conditions does the merged sequence have a lesser order or lesser weight than all contributing sequences?

Example 1. For $i \geq 0$, let $H_i = F_{2i}$ and $J_i = F_{2i+1}$. The merged sequence is the Fibonacci sequence, of order 2 and weight 2, whereas H and J each have order 2 and weight 4, since $H_n = 3H_{n-1} - H_{n-2}$ and $J_n = 3J_{n-1} - J_{n-2}$ This example is generalizable since subsequences whose indices form arithmetic progressions inherit recursivity [1].

Example 2. For $i \geq 0$, let $H_{2i} = F_i$ and $H_{2i+1} = 0$, and similarly, let $J_{2i+1} = F_i$ and $J_{2i} = 0$. The merged sequence, G, satisfies $G_{2i} = G_{2i+1} = F_i$. All three sequences, H, J, and G, satisfy the recursion $K_n = K_{n-2} + K_{n-4}$ of order 4 and hence have the same weight.

Reference. [1] Russell Jay Hendel, "Factorizations of sums of $F(aj-b)$", *The Fibonacci Quarterly*, **45** (2007) 128-133.

Problem 11, posed by Michael Wiener

Given a prime $p > 3$ and $1 < \kappa < p-1$, we call a sequence $(a_n)_n$ in $\mathbb{F}_p$ a Φ_κ-sequence if it is periodic with period $p-1$ and satisfies the linear recurrence $a_n + a_{n+1} = a_{n+\kappa}$ with $a_0 = 1$. Such a sequence is said to be a *complete* Φ_κ*-sequence* if in addition

$$\{a_0, a_1, \ldots, a_{p-2}\} = \{1, \ldots, p-1\}.$$

For instance, every primitive root b mod p generates a complete Φ_κ-sequence $a_n = b^n$ for some (unique) κ. In 1992 Brison [1] proved that for prime.$p > 3$, every complete Fibonacci sequence ($\kappa = 2$) in $\mathbb{F}_p$ is generated by a Fibonacci primitive root (i.e. a root of $x^2 - x - 1$ that is also a primitive root in $\mathbb{F}_p$). In 2007, Gil, Weiner and Zara [2] studied the Padovan case ($\kappa = 3$) and related cases. In particular, they proved that when $x^3 - x - 1$ has fewer than three distinct roots in $\mathbb{F}_p$, then every complete Padovan sequence is generated by a Padovan primitive root. However, in the case of three distinct roots, they proved this result only for certain primes and conjectured that the statement holds for every p.

1. Given a prime $p > 3$, prove that a Φ_3-sequence is complete if and only if $a_n = b^n$, where b is a primitive root in $\mathbb{F}_3$ that satisfies $b^3 = b + 1$.

2. Prove, more generally, that if prime $p > 3$ and any $1 < \kappa < p-1$, then a Φ_κ-sequence is complete if and only if $a_n = b^n$, where b is a primitive root in $\mathbb{F}_p$ that satisfies $b^\kappa = b + 1$.

References.

[1] Brison, Owen, "Complete Fibonacci sequences in finite fields", *The Fibonacci Quarterly* 30 (1992), no. 4, 295-304.

[2] J. Gil, M. Weiner, and C. Zara, "Complete Padovan sequences in finite fields", *The Fibonacci Quarterly* 45 (2007), no. 1, 64–75.

Problem 12, posed by Clark Kimberling

Let S be the set generated by these rules: $1 \in S$, and if $x \in S$, then $2x \in S$ and $1 - x \in S$; so that S grows in generations:

$$g(1) = \{1\},\ g(2) = \{0, 2\},\ g(3) = \{-1, 4\},\ g(4) = \{-3, -2, 8\},\ \ldots$$

Prove or disprove that each generation contains at least one Fibonacci number or its negative.

Problem 13, posed by Marjorie Johnson

Prove that the Fibonacci representations of squares of even subscripted Fibonacci numbers end with 0001; and that the odd subscripted end with 000101. (Hint, consider sums of Fibonacci numbers having subscripts of the form $4j$ or $4j + 2$.) More difficult and more interesting: find all integers M such that M^2 ends in 0.

Problem 14, posed by Ron Knott, solved by Sam Northshield

As an infinite Mancala game, suppose a line of pots contains pebbles, 1 in the first, 2 in the second, and n in the nth, without end. The pebbles are taken from the leftmost non-empty pot and added, one per pot, to the pots to the right. Prove that the number of pebbles in pot n as it is emptied is $\lfloor n\varphi \rfloor$, where φ is the golden ratio, $(1+\sqrt{5})/2$. (This is a variation on a comment by Roland Schroeder on the lower Wythoff sequence; see A000201 in the Online Encyclopedia of Integer Sequences.)

Solution by Sam Northshield. Starting with the positive integers, repeat the following procedure:

* Remove the first entry to create a new row.

* If that number was n, then add 1 to each of the first n entries in the new row, obtaining

1	2	3	4	5	6	7	8	9	10	11	12	...
	3	3	4	5	6	7	8	9	10	11	12	...
		4	5	6	6	7	8	9	10	11	12	...
			6	7	7	8	8	9	10	11	12	...

Lemma.

$$2n - \lfloor n\varphi \rfloor = \lfloor n/\varphi^2 \rfloor = \min\{j : \lfloor j\varphi^2 \rfloor \geq n\}.$$

Proof. The first equality is obvious from $2 - \varphi = 1/\varphi^2$. To prove the second, note that

$$\lfloor n/\varphi^2 \rfloor < n/\varphi^2 \Rightarrow \lfloor n/\varphi^2 \rfloor \varphi^2 < n \Rightarrow \lfloor \lfloor n/\varphi^2 \rfloor \varphi^2 \rfloor < n$$

and

$$n/\varphi^2 < \lceil n/\varphi^2 \rceil \Rightarrow \lceil n/\varphi^2 \rceil \varphi^2 > n \Rightarrow \lfloor \lceil n/\varphi^2 \rceil \varphi^2 \rfloor \geq n.$$

Theorem. *Let d_n denote the first term in the nth row of the array above. Then $d_n = \lfloor n\varphi \rfloor$.*

Proof. We see that the d_{j-1} ones added to the jth row contribute 1 to the value of d_n if $j + d_{j-1} - 1 \geq n$. That is,

$$d_n = n + |\{j \leq n : j + d_{j-1} - 1 \geq n\}|$$

or equivalently,

$$d_n = n + |\{j < n : j + d_{j-1} \geq n\}|$$

Since d_n is strictly increasing, we arrive at the recursive formula

$$d_n = 2n - \min\{j : d_j + j \geq n\},$$

of which the solution is unique (given that $d_1 = 1$) and so it is enough to show that $\lfloor n\varphi \rfloor$ satisfies it; i.e., that

$$\lfloor n\varphi \rfloor = 2n - \min\{j : \lfloor j\varphi \rfloor + j \geq n\}.$$

Since $\lfloor j\varphi \rfloor + j = \lfloor j\varphi^2 \rfloor$, the lemma applies and the proof of the theorem is finished.

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