COMPOSITIONS AND RECURRENCES

WILLIAM WEBB AND NATHAN HAMLIN

Abstract. If a_n denotes the number of compositions of n into parts in a set S, we show that a_n satisfies a recurrence equation if and only if $S = S_1 \cup S_2$ where S_1 is a finite set and $S_2 = \{k \geq k_0 : k \equiv r_1, r_2, ..., r_h \pmod{m}\}$.

1. Introduction

Let a_n denote the number of compositions of n subject to some system of constraints C. If the constraint is using only odd parts, then $a_n = F_n$ (the n^{th} Fibonacci number). Thus, $5 = 3+1+1 = 1+3+1 = 1+1+3 = 1+1+1+1+1$ has $5 = F_5$ compositions. If the constraint is using only parts ≥ 2, then $a_n = F_{n-1}$, and if only parts 1 and 2 are allowed $a_n = F_{n+1}$. All of these are mentioned in the OEIS for the Fibonacci sequence A000045 [10]. In [6] problem 5.4.13 asks to show that a_n is the n^{th} Padovan number, satisfying the recurrence $a_{n+3} = a_{n+1} + a_n$, if only odd parts ≥ 3 are allowed. The Padovan numbers also count the number of compositions into parts congruent to 2 (mod 3). These results are also mentioned in the OEIS for the Padovan sequence A000931.

In some recent papers other constraints on the allowed compositions include: no part of a specified size t [3] [4] [7], at least one part of size t [1], parts of size 1 and t [2], and no parts divisible by 3 [9]. Some of these papers deal with the recurrence satisfied by a_n, others with expressions of a_n as sums of other quantities.

In all the examples described above, except for "at least one part of size t", the type of constraint C is of the form requiring all parts to be chosen from a specified set S. This leads naturally to the question: for which such sets S does a_n satisfy a linear, homogeneous, constant coefficient recurrence equation? Our goal is to answer this question.

2. Generating Functions

We will approach this problem using ordinary generating functions (OGF). One of the key properties of a recurrence sequence a_n is that its OGF is a rational function $P(x)/Q(x)$ where deg $P(x) <$ deg $Q(x)$ [6]. If deg $P(x) \geq$ deg $Q(x)$, then a_n satisfies a recurrence equation except for a finite number of initial terms.

Theorem 2.1. *The number of compositions of n into parts from a set S of positive integers satisfies a linear, homogeneous, constant coefficient recurrence equation, except possibly for finitely many terms, if and only if $S = S_1 \cup S_2$ where S_1 is a finite set, and there are residues $r_1, r_2, ..., r_h$ modulo m such that $S_2 = \{k \geq k_0 : k \equiv r_1, r_2, ..., r_h \pmod{m}\}$.*

Proof. The number of compositions of n into exactly p parts from a set S, is the number of solutions of: $y_1 + y_2 + ... + y_p = n$, $n \geq 1$, where $y_i \in S$.

The corresponding OGF is:

$$f_p(x) = (\sum_{s\in S} x^s)^p = f(x)^p \tag{2.1}$$

If a_n counts the number of such compositions into any number of parts, then its OGF is:

$$\sum_{n\geqslant 0} a_n x^n = \sum_{p=1}^{\infty} f_p(x) = \sum_{p=1}^{\infty} f(x)^p = \frac{f(x)}{1-f(x)} \tag{2.2}$$

Thus, a_n satisfies a recurrence equation if and only if: $\frac{f(x)}{1-f(x)} = \frac{P(x)}{Q(x)}$ is a rational function, which is true if and only if $f(x) = \frac{P(x)}{P(x)+Q(x)}$ is a rational function, which is true if and only if $f(x)$ is the OGF of a sequence which satisfies a recurrence equation except possibly for finitely many terms. Suppose this recurrence equation is of order t. Since all of the coefficients of $f(x)$ are 0 or 1, there are only finitely many different blocks of length t. Hence the coefficients of $f(x)$ must be periodic, but not necessarily purely periodic. That is, S must be of the form described in the theorem. □

3. Some Implications

Theorem 2.1 shows that all of the examples in the introduction satisfy a recurrence equation. Note that the number of compositions with at least one part of size t is the same as all compositions minus those with no parts of size t. The OGF for no parts of size t uses $f(x) = \sum_{i\geqslant 1} x^i - x^t = \frac{x}{1-x} - x^t = \frac{x-x^t+x^{t+1}}{1-x}$, which is a rational function.

Example 1. If a_n counts the number of compositions with no parts of sizes t_1, t_2, ... , t_k, then from the proof of Theorem 2.1, the OGF for a_n is $\frac{f(x)}{1-f(x)}$ where

$$f(x) = \sum_{j\geqslant 1} x^j - x^{t_1} - x^{t_2} - ... - x^{t_k} = \frac{x - x^{t_1} + x^{t_1+1} - x^{t_2} + x^{t_2+1} - ... + x^{t_k+1}}{1-x}. \tag{3.1}$$

Hence, the OGF for a_n is

$$\frac{x - x^{t_1} + x^{t_1+1} - ... + x^{t_k+1}}{1 - 2x + x^{t_1} - x^{t_1+1} + ... - x^{t_k+1}}. \tag{3.2}$$

Thus, a_n satisfies the recurrence equation

$$a_{n+t_k+1} - 2a_{n+t_k} + a_{n+t_k-t_1+1} - ... - a_n = 0. \tag{3.3}$$

Theorem 2.1 also proves that many types of compositions do no satisfy a recurrence equation. For example, the number of compositions into prime numbers, squares, or Fibonacci numbers do not satisfy a recurrence, since these sets are not of the type described in Theorem 2.1. However, compositions into numbers which are either primes or Fibonacci numbers less than 100 are counted by a recurrence since this is a finite set.

Example 2. If b_n counts the number of compositions of n into parts which are congruent to r_1, r_2, ..., r_h modulo m, $0 \le r_i \le m-1$, then $S = \{s\colon\ s \equiv r_1, r_2, ..., r_h \pmod{m}\}$ and

$$f(x) = \sum_{s\in S} x^s = \sum_{i=1}^{h}\sum_{j=0}^{\infty} x^{r_i+jm} = \sum_{i=1}^{h} \frac{x^{r_i}}{1-x^m} = \frac{x^{r_1} + x^{r_2} + ... + x^{r_h}}{1-x^m}. \tag{3.4}$$

By Theorem 2.1, the OGF of the sequence b_n is

$$\frac{f(x)}{1-f(x)} = \frac{x^{r_1} + ... + x^{r_h}}{1 - x^{r_1} - ... - x^{r_h} - x^m}. \tag{3.5}$$

Hence, b_n satisfies the recurrence equation

$$b_{n+m} - b_{n+m-r_1} - ... - b_{n+m-r_h} - b_n = 0. \tag{3.6}$$

Example 3. If c_n counts the number of compositions of n into parts of size 2 or 3 or numbers congruent to 2 or 4 modulo 7 and greater than 14, the function $f(x)$ in Theorem 2.1 is

$$f(x) = x^2 + x^3 + \frac{x^{16} + x^{18}}{1 - x^7}. \tag{3.7}$$

Then the OGF for c_n is

$$\frac{f(x)}{1 - f(x)} = \frac{\frac{x^2+x^3-x^9-x^{10}+x^{16}+x^{18}}{1-x^7}}{\frac{1-x^7-x^2-x^3+x^9+x^{10}-x^{16}-x^{18}}{1-x^7}} = \frac{x^2 + x^3 - x^9 - x^{10} + x^{16} + x^{18}}{1 - x^2 - x^3 - x^7 + x^9 + x^{10} - x^{16} - x^{18}}. \tag{3.8}$$

Thus, c_n satisfies the recurrence

$$c_{n+18} - c_{n+16} - c_{n+15} - c_{n+11} + c_{n+9} + c_{n+8} - c_{n+2} - c_n = 0. \tag{3.9}$$

Suppose we are given a recurrence sequence a_n and ask if there is a type of composition which is counted by a_n. As in Theorem 2.1 if $\sum_{n\geqslant 0} a_n x^n = \frac{P(x)}{Q(x)}$ then $\sum_{s\in S} x^s = f(x) = \frac{P(x)}{P(x)+Q(x)}$.

Example 4. Is there a composition counted by the Fibonacci sequence so that $a_n = F_n$ for $n \geq 1$? Since $\sum_{n\geqslant 1} F_n x^n = \frac{x}{1-x-x^2} = \frac{P(x)}{Q(x)}$, $f(x) = \frac{x}{1-x^2} = \sum_{i\geqslant 0} x^{2i+1}$. Hence, a_n counts compositions into odd parts. Similarly, if we want $a_n = F_{n+1}$, since $\sum_{n\geqslant 1} F_{n+1} x^n = \frac{x+x^2}{1-x-x^2} = \frac{P(x)}{Q(x)}$, $f(x) = x + x^2$ so a_n counts compositions into parts of size 1 or 2. If $a_n = F_{n-1}$, since $\sum_{n\geqslant 1} F_{n-1} x^n = \frac{x^2}{1-x-x^2} = \frac{P(x)}{Q(x)}$, $f(x) = \frac{x^2}{1-x} = \sum_{n\geqslant 2} x^n$ so a_n counts compositions into parts greater than or equal to 2. However, for $a_n = F_{n+2}$, a similar calculation leads to $f(x) = \frac{2x+x^2}{1+x} = 2x - x^2 + x^3 - x^4 + \cdots$, which is not of the form $\sum_{s\in S} x^s$.

Example 5. Is there a composition counted by the tribonacci sequence? In this case the sequence satisfies the recurrence equation $a_{n+3} - a_{n+2} - a_{n+1} - a_n = 0$ with the usual initial values $a_1 = 0$, $a_2 = 1$, $a_3 = 1$. The OGF is $\frac{x^2}{1-x-x^2-x^3}$ so $f(x) = \frac{x^2}{1-x-x^3} = x^2 + x^3 + x^4 + 2x^5 + 3x^6 + \cdots$. Since this series has coefficients other that 0 or 1 it cannot equal $\sum_{s\in S} x^s$. Thus, there is no composition of the desired kind. However, if we change the initial values but keep the tribonacci recurrence, so that the OGF is $\frac{x+x^2}{1-x-x^2-x^3}$, i.e., $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, then $f(x) = \frac{x+x^2}{1-x^3} = (x + x^2)\sum_{i\geqslant 0} x^{3i} = \sum_{i\geqslant 0}(x^{3i+1} + x^{3i+2})$ so S is the set of positive integers congruent to 1 or 2 (mod 3).

There are other types of constraints that are not of the kind described in Theorem 2.1, such as restricting the number of times specific parts can be used, or if the choice for one part restricts the choice for another part. We hope to address compositions of such types in the future.

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Department of Mathematics, Washington State University, Pullman, Washington, 99163, USA
E-mail address: `webb@math.wsu.edu`

Department of Mathematics, Washington State University, Pullman, Washington, 99163, USA
E-mail address: `nghamlin@wsu.edu`