SYMMETRIES OF STIRLING NUMBER SERIES

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ABSTRACT. We consider Dirichlet series generated by weighted Stirling numbers, focusing on a symmetry of such series which is reminiscent of a duality relation of negative-order poly-Bernoulli numbers. These series are connected to several types of zeta functions and this symmetry plays a prominent role. We do not know whether there are combinatorial explanations for this symmetry, as there are for the related poly-Bernoulli identity.

1. Introduction

This paper is concerned with the Dirichlet series

$$S_{j,r}(s,a) = \sum_{m=j}^{\infty} \frac{(-1)^{m+j} s(m,j|r)}{m!(m+a)^s} \tag{1.1}$$

where $s(m,j|r)$ denotes the *weighted Stirling number of the first kind* [4, 5] defined for nonnegative integers m, j and $r \in \mathbb{C}$ by the vertical generating function

$$(1+t)^{-r}(\log(1+t))^j = j! \sum_{m=j}^{\infty} s(m,j|r) \frac{t^m}{m!} \tag{1.2}$$

or by the horizontal generating function

$$(x)_m = \sum_{j=0}^{m} s(m,j|r)(x+r)^j \tag{1.3}$$

where $(x)_m = x(x-1)\cdots(x-m+1)$ denotes the falling factorial. If j is a nonnegative integer, $S_{j,r}(s,a)$ converges for $r,s,a \in \mathbb{C}$ such that $\Re(s) > \Re(r)$ and $\Re(a) > -j$; when $r \in \mathbb{Z}^+$ it has poles of order $j+1$ at $s = 1,2,..,r$ and of order at most j at nonpositive integers s. When $j = 0$ we recover the *Barnes multiple zeta functions*, and when $j = 1$ we obtain special values of *non-strict multiple zeta functions*, also known as *zeta-star values* (see section 3). We will focus on the symmetric identity

$$S_{j,r}(k+1, 1-t) = S_{k,t}(j+1, 1-r), \tag{1.4}$$

valid for integers $r \le k$ and $t \le j$, which bears a striking resemblance to a symmetric identity of *poly-Bernoulli polynomials* (Theorem 6.1 below). Since this poly-Bernoulli identity has known combinatorial interpretations in the case where $r = t = 0$, we find it interesting to ask whether the symmetry (1.4) may be proved or interpreted in terms of counting arguments.

2. Stirling and r-Stirling numbers

The weighted Stirling numbers of the first kind $s(n,k|r)$ may be defined by either (1.2) or (1.3), or by the recursion

$$s(n+1,k|r) = s(n,k-1|r) - (n+r)s(n,k|r) \tag{2.1}$$

with initial conditions $s(n,n|r) = 1$, $s(n,0|r) = (-r)_n$. Their dual companions [8] are the *weighted Stirling numbers of the second kind* $S(n,k|r)$ [4, 5] which may be defined by the vertical generating function

$$e^{rt}(e^t-1)^m = m! \sum_{n=m}^{\infty} S(n,m|r)\frac{t^n}{n!}, \tag{2.2}$$

the horizontal generating function

$$x^n = \sum_{k=0}^{n} S(n,k|r)(x-r)_k, \tag{2.3}$$

or by the recursion

$$S(n+1,k|r) = S(n,k-1|r) + (k+r)S(n,k|r) \tag{2.4}$$

with initial conditions $S(n,n|r) = 1$, $S(n,0|r) = r^n$. It is clear that both $s(n,k|r)$ and $S(n,k|r)$ are polynomials in r with integer coefficients of degree $n-k$ whose derivatives are given by

$$s'(n,k|r) = (k+1)s(n,k+1|r) \qquad \text{and} \qquad S'(n,k|r) = nS(n-1,k|r). \tag{2.5}$$

For combinatorial interpretations, when the "weight" r is a nonnegative integer we may write

$$(-1)^{m+j}s(m,j|r) = \begin{bmatrix} m+r \\ j+r \end{bmatrix}_r \tag{2.6}$$

in terms of *r-Stirling numbers* $\begin{bmatrix} n \\ k \end{bmatrix}_r$, which count the number of permutations of $\{1,2,...,n\}$ having k cycles, with the elements $1,2,...,r$ restricted to appear in different cycles [3, 1]. When $r=0$ these definitions reduce to those of the usual Stirling numbers, and in that case the parameter r is often suppressed in the notation. Furthermore if $j=1$ and $r \geq 0$ the coefficients $(-1)^{m+1}s(m,1|r)/m!$ are called *hyperharmonic numbers* $H_m^{[r]}$ defined by $H_m^{[0]} = \frac{1}{m}$ for $m > 0$, $H_0^{[r]} = 0$, and

$$H_m^{[r]} = \sum_{i=1}^{m} H_i^{[r-1]} \tag{2.7}$$

(cf. [1, 14, 9]). Thus $H_n = H_n^{[1]}$ denotes the usual harmonic number.

3. Dirichlet series Identities

Our interest in the series (1.1) is derived from the fact that they specialize to known multiple zeta functions when $j = 0,1$. First, the series $S_{0,1}(s,1)$ is the Riemann zeta function $\zeta(s)$; more generally for $r \in \mathbb{Z}^+$ the series $S_{0,r}(s,a)$ is a *Barnes multiple zeta function* $\zeta_r(s,a)$ [15, 16] defined for $\Re(s) > r$ and $\Re(a) > 0$ by

$$\zeta_r(s,a) = \sum_{t_1=0}^{\infty} \cdots \sum_{t_r=0}^{\infty} (a+t_1+\cdots+t_r)^{-s}. \tag{3.1}$$

If we view $\zeta_r(s,a)$ as an analytic function of its order r as in [15, 16], then we can view $S_{j,r}(s,a) = j!D_r^j\zeta_r(s,a)$ by means of (2.5), where D_r denotes the derivative d/dr. From this identification we deduce from ([16], Corollary 2) that the series $S_{j,r}(s,a)$ is convergent when $\Re(s) > \Re(r)$ and $\Re(a) > -j$.

For $r \in \mathbb{Z}^+$ the series $S_{1,r}(s,0)$ is also a specialization of a *non-strict multiple zeta function*, namely $S_{1,r}(s,0) = \zeta^\star(s, \underbrace{0,...,0}_{r-1}, 1)$, where

$$\zeta^\star(s_1,...,s_m) := \sum_{n_1 \geq n_2 \geq \cdots \geq n_m \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_m^{s_m}} \tag{3.2}$$

([9], Prop. 2.1). The zeta-star values are related to Arakawa-Kaneko zeta functions, whose values at negative integers are given by the poly-Bernoulli numbers $\mathbb{B}_n^{(k)}$ ([9, 6]).

The series (1.1) satisfies several identities.

Theorem 3.1. *The following identities hold where defined.*

i. *We have* $S_{j,r}(s,a) = S_{j,r}(s,a+1) + S_{j,r-1}(s,a)$.
ii. *For* $r \in \mathbb{Z}^+$ *we have* $S_{j,r}(s,a) = S_{j,0}(s,a) + \sum_{t=1}^r S_{j,t}(s,a+1)$.
iii. *For* $0 \leq m \leq r$ *we have* $S_{j,r}(s,a) = \sum_{t=0}^m \binom{m}{t} S_{j,r-t}(s,a+m-t)$.
iv. *We have*

$$S_{j,r}(s,a) - aS_{j,r}(s+1,a) = S_{j-1,r+1}(s+1,a+1) + rS_{j,r+1}(s+1,a+1).$$

v. *(Symmetry relation.) For integers* $r \leq k$ *and* $t \leq j$ *we have*

$$S_{j,r}(k+1, 1-t) = S_{k,t}(j+1, 1-r).$$

Thus when it converges, the series $S_{j,r}(k+1,1-t)$ *is invariant under* $(j,k,r,t) \mapsto (k,j,t,r)$.

Proof. Identity (i) follows from the Stirling number recurrence (2.1), or equivalently from the difference equation

$$\zeta_r(s,a) - \zeta_r(s,a+1) = \zeta_{r-1}(s,a) \tag{3.3}$$

([15], eq. (2.1)) of the Barnes multiple zeta functions. Identities (ii) and (iii) may be obtained by induction from (i), or from Identity 5 and Identity 7 in [1]. To obtain (iv), we differentiate the generating function (1.2) with respect to r and equate coefficients of $t^n/n!$ to obtain

$$s(n+1,j|r) = s(n,j-1|r+1) - r\, s(n,j|r+1). \tag{3.4}$$

Dividing by $(n+1)!(n+a)^s$ and summing over n then yields (iv). By means of (2.5) we have $S_{j,r}(s,a) = j! D_r^j \zeta_r(s,a)$, and therefore the symmetry relation (v) follows from the identity

$$(k-1)! D_t^{j-1} \zeta_t(k, 1-r) = (j-1)! D_r^{k-1} \zeta_r(j, 1-t) \tag{3.5}$$

([16], Corollary 2). □

4. Combinatorial interpretation

Restricting our attention to the case where r is a nonnegative integer, the symmetry relation Theorem 3.1(v) may be written as

$$\sum_{m=j}^{\infty} \frac{\begin{bmatrix} m+r \\ j+r \end{bmatrix}_r}{m!(m+1-t)^{k+1}} = \sum_{m=k}^{\infty} \frac{\begin{bmatrix} m+t \\ k+t \end{bmatrix}_t}{m!(m+1-r)^{j+1}} \tag{4.1}$$

for integers $0 \leq r \leq k$ and $0 \leq t \leq j$, where the r-Stirling number $\begin{bmatrix} n \\ k \end{bmatrix}_r =$ the number of permutations of $\{1,2,...,n\}$ having k cycles, with the elements $1,2,...,r$ restricted to appear in different cycles. When $r, t \in \{0,1\}$ this gives series identities for the usual Stirling numbers of the first kind; for example, in

$$\sum_{m=j}^{\infty} \frac{\begin{bmatrix} m \\ j \end{bmatrix}}{m!(m+1)^{k+1}} = \sum_{m=k}^{\infty} \frac{\begin{bmatrix} m \\ k \end{bmatrix}}{m!(m+1)^{j+1}} \tag{4.2}$$

we have $\left[{m \atop k}\right]/m!$ equal to the proportion of permutations of $\{1, ..., m\}$ which have k cycles. Thus the left side of (4.2) may be viewed as a sum over permutations which have j cycles and the right side as a sum over permutations which have k cycles.

Question 1: Can the identities (4.2) or (4.1) be proved by combinatorial means?

5. Values at positive integers

The identities of section 3 may be used to demonstrate a large class of values of $S_{j,r}(s,a)$ which may be expressed as polynomials in values of the Riemann zeta function.

Theorem 5.1. *When $j \in \{0,1\}$ or $s \in \{1,2\}$ we have $S_{j,r}(s,a) \in \mathbb{Q}[\zeta(2), \zeta(3), \zeta(5), ...]$ for integers $r < s$ and $a > -j$.*

Proof. Write $R = \mathbb{Q}[\zeta(2), \zeta(3), \zeta(5), ...]$. When $j = 0$ and $r \leq 0$ the sum for $S_{j,r}(s,a)$ is finite, and therefore rational, so the theorem is therefore true in that case. For $j = 0$ and $r > 0$ we have $S_{0,r}(s,a) = \zeta_r(s,a)$ and we use the identity

$$\zeta_r(s,a) = \frac{1}{(r-1)!} \sum_{k=0}^{r-1} s(r-1, k|a+1-r)\, \zeta_1(s-k, a) \tag{5.1}$$

([16], eq. (3.3)) to prove the theorem in that case, since $\zeta_1(s,a) \in R$ for integers $s > 1$ and $a > 0$. The theorem is therefore established for $j = 0$.

In the case $j = 1$ the theorem generalizes Euler's classical identity

$$S_{1,1}(s,0) = \sum_{n=1}^{\infty} \frac{H_n}{n^s} = \frac{s+2}{2}\zeta(s+1) - \frac{1}{2}\sum_{j=1}^{s-2} \zeta(s-j)\zeta(j+1) \in R. \tag{5.2}$$

Kamano [9] proved that

$$(r-1)! S_{1,r}(s,0) = \sum_{k=1}^{r} \begin{bmatrix} r \\ k \end{bmatrix} S_{1,1}(s,0) + \left(k \begin{bmatrix} r \\ k+1 \end{bmatrix} - \begin{bmatrix} r \\ k \end{bmatrix} H_{r-1} \right) \zeta(s+1-k) \tag{5.3}$$

which, together with (5.2), implies that $S_{1,r}(s,0) \in R$ when $r > 0$. (Alternatively one can use the recursion

$$S_{1,r}(s,0) = S_{1,1}(s,0) + \sum_{k=1}^{r-1} \frac{1}{k} \left(S_{1,k}(s-1,0) + B(k,s) \right) \tag{5.4}$$

([14], Theorem 6), where $B(k,s)$ is a linear polynomial in $\{\zeta(j)\}_{m \geq 2}$, to show this). When $j = 1$ and $r = 0$ we observe that $S_{1,0}(1,a) = H_a/a \in \mathbb{Q}$ for $a \in \mathbb{Z}^+$; induction using Theorem 3.1(iv) then shows $S_{1,0}(s,a) \in R$ for all $s > r$ and $a \geq 0$. So $S_{1,r}(s,a) \in R$ when either $a = 0$ or $r = 0$; an induction argument using Theorem 3.1(i) shows that $S_{1,r}(s,a) \in R$ when $r \geq 0$ and $a \geq 0$.

A similar induction argument, using Theorem 3.1(i) and (iv), shows that $S_{1,r}(s,a) \in R$ for $a \geq 0$ when r is a negative integer and $s > r$. This completes the proof of the theorem for

$j \in \{0, 1\}$. The statement concerning $s \in \{1, 2\}$ then is obtained by the symmetry relation Theorem 3.1(v). □

6. Poly-Bernoulli polynomials

In this final section we prove a finite sum symmetric identity which bears a striking resemblance to the infinite sum symmetric identity of Theorem 3.1(v). The *weighted shifted poly-Bernoulli numbers* $\mathbb{B}_n^{(k)}(a, r)$ of order k are defined by

$$\Phi(1 - e^{-t}, k, a)e^{-rt} = \sum_{n=0}^{\infty} \mathbb{B}_n^{(k)}(a, r)\frac{t^n}{n!} \tag{6.1}$$

$$\text{where} \qquad \Phi(z, s, a) = \sum_{m=0}^{\infty} \frac{z^m}{(m + a)^s} \qquad\qquad (|z| < 1) \tag{6.2}$$

is the Lerch transcendent. (The generalization (6.1) was communicated to me by Mehmet Cenkci, to whom I am grateful). When $a = 1$ and $r = 0$ we obtain the usual poly-Bernoulli numbers $\mathbb{B}_n^{(k)} = \mathbb{B}_n^{(k)}(1, 0)$ defined and studied by Kaneko [10], since in that case the Lerch transcendent reduces to the usual order k polylogarithm function

$$\mathrm{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k}. \tag{6.3}$$

The $\mathbb{B}_n^{(k)}(a, r)$ are polynomials of degree n in r and they are polynomials of degree $-k$ in a when $-k \in \mathbb{Z}^+$. When $k = 1$ and $a = 0$ we have

$$\mathbb{B}_n^{(1)}(0, r) = (-1)^n B_n(r) \tag{6.4}$$

in terms of the usual Bernoulli polynomials $B_n(x)$. The weighted Lerch poly-Bernoulli numbers may also be expressed in terms of weighted Stirling numbers of the second kind as

$$\mathbb{B}_n^{(k)}(a, r) = (-1)^n \sum_{m=0}^{n} \frac{(-1)^m m! S(n, m|r)}{(m + a)^k}. \tag{6.5}$$

Therefore in the case $r = 0$ these polynomials agree with the shifted poly-Bernoulli numbers of ([12], §6). The weighted shifted poly-Bernoulli polynomials satisfy the following symmetric identity.

Theorem 6.1. *For all nonnegative integers n and k we have*

$$\mathbb{B}_n^{(-k)}(1 - t, r) = \mathbb{B}_k^{(-n)}(1 - r, t).$$

Proof. This result was proved by Kaneko [10] in the case $r = 0$, $t = 0$, and the proof is adapted from Kaneko's proof. Straightforward calculation shows that

$$\begin{aligned}
\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\mathbb{B}_n^{(-k)}(1-a,x)\frac{t^n}{n!}\frac{u^k}{k!} &= \sum_{k=0}^{\infty}\Phi(1-e^{-t},-k,1-a)e^{-xt}\frac{u^k}{k!} \\
&= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(1-e^{-t})^m e^{-xt}u^k}{(m+1-a)^{-k}k!} \\
&= e^{-xt}\sum_{m=0}^{\infty}(1-e^{-t})^m e^{(m+1-a)u} \\
&= e^{-xt}e^{(1-a)u}\sum_{m=0}^{\infty}((1-e^{-t})e^u)^m \\
&= \frac{e^{-xt}e^{(1-a)u}}{1-(1-e^{-t})e^u} \\
&= \frac{e^{(1-x)t}e^{(1-a)u}}{e^t+e^u-e^{t+u}} \qquad (6.6)
\end{aligned}$$

is invariant under $(t,u,a,x)\mapsto(u,t,x,a)$.

□

This theorem says that the expression $\mathbb{B}_n^{(-k)}(1-t,r)$ is a polynomial in r and t which is invariant under $(n,k,r,t)\mapsto(k,n,t,r)$. In terms of weighted Stirling numbers it reads

$$\sum_{m=0}^{n}(-1)^{m+n}m!S(n,m|r)(m+1-t)^k = \sum_{m=0}^{k}(-1)^{m+k}m!S(k,m|t)(m+1-r)^n. \qquad (6.7)$$

We find this identity to be strikingly similar to the symmetric identity, for $r \le k$ and $t \le j$,

$$\sum_{m=j}^{\infty}\frac{(-1)^{m+j}s(m,j|r)}{m!(m+1-t)^{k+1}} = \sum_{m=k}^{\infty}\frac{(-1)^{m+k}s(m,k|t)}{m!(m+1-r)^{j+1}}, \qquad (6.8)$$

given by Theorem 3.1(v). The two identities appear to share a kind of duality, but it is curious that one identity is for finite sums and the other is for infinite series.

In the case $r = t = 0$, the poly-Bernoulli numbers $\mathbb{B}_n^{(-k)}$ have found at least two important combinatorial interpretations. In [2] it is shown that $\mathbb{B}_n^{(-k)}$ equals the number of distinct $n\times k$ lonesum matrices, where a *lonesum matrix* is a matrix with entries in $\{0,1\}$ which is uniquely determined by its row and column sums. In [13] it is shown that the number of permutations σ of the set $\{1,2,...,n+k\}$ which satisfy $-k \le \sigma(i)-i \le n$ for all i is the poly-Bernoulli number $\mathbb{B}_n^{(-k)}$. Either of these two combinatorial interpretations make the $r = t = 0$ case of the symmetry relation of Theorem 6.1 obvious.

Question 2. Can the symmetric identity of Theorem 6.1 be proved by a counting argument in cases where r and t are nonzero integers?

References

[1] A. Benjamin, D. Gaebler, R. Gaebler, *A combinatorial approach to hyperharmonic numbers*, INTEGERS **3** (2003), article A15, 9pp.

[2] C. Brewbaker, *A combinatorial interpretation of the poly-Bernoulli numbers and two Fermat analogues*, INTEGERS **8** (2008), article A02, 9pp.

[3] A. Broder, *The r-Stirling numbers*, Disc. Math. **49** (1984), 241-259.

[4] L. Carlitz, *Weighted Stirling numbers of the first and second kind - I*, The Fibonacci Quarterly **18** (1980), 147-162.

[5] L. Carlitz, *Weighted Stirling numbers of the first and second kind - II*, The Fibonacci Quarterly **18** (1980), 242-257.

[6] M.-A. Coppo and B. Candelpergher, *The Arakawa-Kaneko zeta functions*, Ramanujan J. **22** (2010), 153-162.

[7] F. T. Howard, *Congruences and recurrences for Bernoulli numbers of higher order*, The Fibonacci Quarterly **32** (1994), 316-328.

[8] L. Hsu and P. Shiue, *A unified approach to generalized Stirling numbers*, Adv. Appl. Math. **20** (1998), 366-384.

[9] K. Kamano, *Dirichlet series associated with hyperharmonic numbers*, Mem. Osaka Inst. Tech. Ser. A **56** (2011), 11-15.

[10] M. Kaneko, *Poly-Bernoulli numbers*, J. Théorie Nombres Bordeaux **9** (1997), 199-206.

[11] M. Kaneko, *A note on poly-Bernoulli numbers and multiple zeta values*, in: Diophantine Analysis and Related Fields, AIP Conference Proceedings vol. 976, 118-124, AIP, New York, 2008.

[12] T. Komatsu and L. Szalay, *Shifted poly-Cauchy numbers*, Lithuanian Math. J. **54** (2014), 166-181.

[13] S. Launois, *Rank t $\mathcal{H}$-primes in quantum matrices*, Comm. Alg. **33** (2005), 837-854.

[14] I. Mezö and A. Dil, *Hyperharmonic series involving Hurwitz zeta function*, J. Number Theory **130** (2010), 360-369.

[15] P. T. Young, *Rational series for multiple zeta and log gamma functions*, J. Number Theory **133** (2013), 3995-4009.

[16] P. T. Young, *Symmetries of Bernoulli polynomial series and Arakawa-Kaneko zeta functions*, J. Number Theory **143** (2014), 142-161.

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