A BIJECTIVE PROOF OF A DERANGEMENT RECURRENCE

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ABSTRACT. The number of permutations of order n with no fixed points is called the nth derangement number, and is denoted by D_n. It is well-known that for $n > 1$, the derangement numbers satisfy the recurrence $D_n = nD_{n-1}+(-1)^n$. We present a simple combinatorial proof of this recurrence.

Let D_n denote the set of *derangements* on n elements, the number of permutations of $\{1, \ldots, n\}$ with no fixed points. Using cycle notation, we see that $D_1 = 0$, $D_2 = 1$ counts (12), $D_3 = 2$ counts (123) and (132), $D_4 = 9$ counts

$$(12)(34), (13)(24), (14)(23), (1234), (1243), (1324), (1342), (1423), (1432)$$

and so on. The well-known principle of inclusion-exclusion arrives at a closed form for D_n. Specifically, for $n \geq 1$, $D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$. From this, it follows that for $n > 1$,

$$D_n = nD_{n-1} + (-1)^n. \tag{0.1}$$

Richard Stanley points out in his classic text [2] that "considerably more work is required to prove (0.1) combinatorially" and refers to articles by Remmel [1] and Wilf [3]. In this note, we present another simple bijective proof, where we create an almost-1-to-1 correspondence between F, the set of permutations of n elements with exactly one fixed point and D, the set of derangements of n elements. Naturally, $|D| = D_n$ and $|F| = nD_{n-1}$. The word "almost" reflects the fact that there will either be one unmapped element of F or one unhit element of D, depending on the parity of n.

We shall express all permutations using cycle notation with the convention that each cycle begins with its smallest element and the cycles are listed in increasing order of its smallest element, such as (164)(25)(3). Hence the first cycle will always begin with element 1. We first describe our bijection between those elements of F and D, when we exclude from both sets those permutations that contain the 2-cycle (12). For permutation π in F, let α denote the fixed point of π.

Our bijection is based on the number of elements in the first cycle of π, and we consider four cases. Case I: When the first cycle has three or more elements, so it looks like $(1\, a_1\, a_2\, \cdots a_j)$ where $j \geq 2$, then π is mapped to the derangement where 1 is mapped to α, α is mapped to a_1, and all other elements are mapped to the same elements as before. We illustrate this by

$$(1\, a_1\, a_2\, \cdots a_j) \cdots (\alpha) \cdots \longrightarrow (1\, \alpha\, a_1\, a_2\, \cdots a_j) \cdots .$$

For example, (164)(25)(3) is mapped to (1364)(25). This mapping hits every derangement where the first cycle has four or more elements. Case II: When the first cycle has two elements, so it looks like $(1\, a_1)$, where $a_1 \neq 2$, then we proceed as before:

$$(1\, a_1) \cdots (\alpha) \cdots \longrightarrow (1\, \alpha\, a_1) \cdots .$$

This mapping hits every derangement where the first cycle is a 3-cycle of the form $(1\, x\, y)$ where $y \neq 2$. To reach 3-cycles of the form $(1\, x\, 2)$ we use permutations from Case III, where

π begins $(1)(2\,a_1)$. Here, we have

$$(1)(2\,a_1)\cdots \longrightarrow (1\,a_1\,2)\cdots .$$

Finally, Case IV maps the permutations where 1 is a fixed point and the second cycle has at least three elements, so $\pi = (1)(2\,a_1\,a_2\,\cdots a_j)$ where $j \geq 2$. These permutations are mapped by

$$(1)(2\,a_1\,a_2\,\cdots a_j)\cdots \longrightarrow (1\,a_1)(2\,a_2\,\cdots a_j)\cdots ,$$

which hits all derangements beginning with a 2-cycle $(1\,x)$ where $x \neq 2$.

It remains to describe the bijection for the permutations in F and D that begin with cycle $(1\,2)$. Let $\sigma_k = (1\,2)(3\,4)\cdots(2k-1,\, 2k)$. We say that a permutation has type k if it begins with σ_k, but does not begin with σ_{k+1}. For example, $(1\,2)(3\,4)(5\,7)(6)$ has type 2. For $1 \leq k \leq (n-3)/2$, there is a bijection between the type k permutations in F and the type k derangements in D. The bijection is essentially the same as before, where elements 1 and 2 are replaced with elements $2k+1$ and $2k+2$, respectively, as below.

$$\sigma_k(2k+1\,a_1\,a_2\,\cdots a_j)\cdots(\alpha)\cdots \longrightarrow \sigma_k(2k+1\,\alpha\,a_1\,a_2\,\cdots a_j)\cdots$$
$$\sigma_k(2k+1\,a_1)\cdots(\alpha)\cdots \longrightarrow \sigma_k(2k+1\,\alpha\,a_1)\cdots$$
$$\sigma_k(2k+1)(2k+2\,a_1)\cdots \longrightarrow \sigma_k(2k+1\,a_1\,2k+2)\cdots$$
$$\sigma_k(2k+1)(2k+2\,a_1\,a_2\,\cdots a_j)\cdots \longrightarrow \sigma_k(2k+1\,a_1)(2k+2\,a_2\,\cdots a_j)\cdots .$$

Notice that when n is even, there is only one permutation of type $n/2-1$, which is the permutation $(12)(34)\cdots(n-3,\, n-2)(n-1)(n)$. But since it has two fixed points, it is not in F or D. There is one permutation of type $n/2$, namely $\sigma_{n/2} = (12)(34)\cdots(n-3,\, n-2)(n-1,\, n)$, which belongs to D but not F. Thus when n is even, $D_n = nD_{n-1} + 1$. Likewise, when n is odd, there is just one permutation of type $(n-1)/2$, namely $(12)(34)\cdots(n-2,\, n-1)(n)$, which belongs to F but not D. Thus when n is odd, $D_n = nD_{n-1} - 1$, as desired.

References

[1] J. B. Remmel, A Note on a Recursion for the Number of Derangements, *European J. Combinatorics*, **4** (1983), 371–374.

[2] R. P. Stanley, *Enumerative Combinatorics*, Volume I, Cambridge University Press, Cambridge, 1986.

[3] H. S. Wilf, A Bijection in the Theory of Derangements, *Mathematics Magazine*, **57** (1984), 37–40.

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