WEIGHTED SUMS OF SQUARES VIA GENERALIZED EULERIAN POLYNOMIALS

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Abstract. We give explicit formulas for the weighted sum of squares $\sum_{j=0}^{m-1} z^j (aj+b)^2$, where $a, b \in \mathbb{C}$ are given, and $z \in \mathbb{C}$, $z \neq 0, 1$ is the weight. In the case $a, b \in \mathbb{Z}$ and $z \in \mathbb{Q}$, we show that there is a one-to-one correspondence between our weighted sums and Primitive Pythagorean Triples. The main tools we use are the Z-transform of sequences and a generalization of Eulerian polynomials.

1. Introduction

Eulerian polynomials 1, $z+1$, z^2+4z+1, $z^3+11z^2+11z+1, \ldots$ were originally considered by L. Euler ([3], pp. 485,486). The corresponding coefficients $A(p,i)$ are the Eulerian numbers, and commonly one finds these numbers in a triangular array (Eulerian numbers triangle), with $p = 1, 2, \ldots$ (rows), and $i = 0, 1, 2, \ldots$ (columns):

$p \diagdown i$	0	1	2	3	4		$\cdots$
1	0	1					
2	0	1	1				$\cdots$
3	0	1	4	1			
4	0	1	11	11	1		$\cdots$
5	0	1	26	66	26	1	
$\vdots$		$\vdots$		$\vdots$		$\vdots$	

(Some shifted versions of this triangle appear in the literature also as Eulerian numbers triangle.) We will write $\mathbb{P}_p(z)$ to denote the Eulerian polynomial $\sum_{i=0}^{p} A(p,i) z^{p-i}$, where $p \in \mathbb{N}$. Observe that $\mathbb{P}_p(z)$ is a $(p-1)$-th degree polynomial.

We will work with the *Z-transform* of sequences, which is a map $\mathcal{Z}$ that takes complex sequences $a_n = (a_0, a_1, \ldots, a_n, \ldots)$ into complex functions $\mathcal{Z}(a_n)(z)$ (or simply $\mathcal{Z}(a_n)$) given by the Laurent series $\mathcal{Z}(a_n) = \sum_{n=0}^{\infty} \frac{a_n}{z^n}$ (defined for $|z| > R^{-1}$, where $R > 0$ is the radius of convergence of the Taylor series $\sum_{n=0}^{\infty} a_n z^n$, the generating function of the sequence a_n. In particular, in any discussion of Z-transform, it is always assumed that $z \neq 0$). If $\mathcal{Z}(a_n) = A(z)$, we also say that the sequence a_n is the *inverse Z-transform* of the complex function $A(z)$, and we write $a_n = \mathcal{Z}^{-1}(A(z))$. We will recall now some basic facts about the Z transform, that we will use throughout the work (for further reading, see [4], [8]).

The sequence λ^n (where λ is a given non-zero complex number), has Z-transform

$$\mathcal{Z}(\lambda^n) = \sum_{n=0}^{\infty} \frac{\lambda^n}{z^n} = \frac{1}{1-\frac{\lambda}{z}} = \frac{z}{z-\lambda}, \tag{1.1}$$

defined for $|z| > |\lambda|$. In particular, the Z-transform of the constant sequence 1 and the alternating sequence $(-1)^n$ are

$$\mathcal{Z}(1) = \frac{z}{z-1} \qquad \text{and} \qquad \mathcal{Z}((-1)^n) = \frac{z}{z+1}, \tag{1.2}$$

respectively.

Three important properties of the Z-transform (which formal proofs are easy exercises left to the reader), are the following:

(1) $\mathcal{Z}$ is linear and injective.
(2) (Advance-shifting property) If $\mathcal{Z}(a_n) = \mathcal{A}(z)$, and $k \in \mathbb{N}$ is given, then

$$\mathcal{Z}(a_{n+k}) = z^k \left(\mathcal{A}(z) - \sum_{j=0}^{k-1} \frac{a_j}{z^j} \right). \tag{1.3}$$

(3) (Multiplication by the sequence n) If $\mathcal{Z}(a_n) = \mathcal{A}(z)$, then

$$\mathcal{Z}(na_n) = -z \frac{d}{dz} \mathcal{A}(z). \tag{1.4}$$

In particular, by using (1.2) and (1.4), we have

$$\begin{aligned} \mathcal{Z}(n) &= -z \frac{d}{dz} \frac{z}{z-1} = \frac{z}{(z-1)^2}, \\ \mathcal{Z}\left(n^2\right) &= -z \frac{d}{dz} \frac{z}{(z-1)^2} = \frac{z(z+1)}{(z-1)^3}, \\ \mathcal{Z}\left(n^3\right) &= -z \frac{d}{dz} \frac{z(z+1)}{(z-1)^3} = \frac{z\left(z^2+4z+1\right)}{(z-1)^4}, \end{aligned} \tag{1.5}$$

and so on.

We can recognize the numerators of (1.5) as (z times) the Eulerian polynomials $\mathbb{P}_1(z) = 1$, $\mathbb{P}_2(z) = z + 1$, $\mathbb{P}_3(z) = z^2 + 4z + 1$. In fact, the affirmation

$$\mathcal{Z}(n^p) = \frac{z\mathbb{P}_p(z)}{(z-1)^{p+1}},$$

is equivalent to the formula $\mathbb{P}_p(z) = z(1-z)\mathbb{P}'_{p-1}(z) + (1 + (p-1)z)\mathbb{P}_{p-1}(z)$, which is the well-known recurrence for Eulerian polynomials.

Observe that given $a, b \in \mathbb{C}$, we have

$$\begin{aligned} \mathcal{Z}\left((an+b)^2\right) &= a^2 \mathcal{Z}\left(n^2\right) + 2ab\mathcal{Z}(n) + b^2\mathcal{Z}(1) \\ &= a^2 \frac{z(z+1)}{(z-1)^3} + 2ab \frac{z}{(z-1)^2} + b^2 \frac{z}{z-1} \\ &= z \frac{b^2z^2 + \left(a^2 + 2ab - 2b^2\right)z + (a-b)^2}{(z-1)^3}. \end{aligned} \tag{1.6}$$

The polynomial

$$\mathbb{P}_{a,b,2}(z) = b^2z^2 + \left(a^2 + 2ab - 2b^2\right)z + (a-b)^2, \tag{1.7}$$

is the *generalized Eulerian polynomial* that appears in the (numerator of the) Z-transform of the sequence $(an+b)^2$, namely

$$\mathcal{Z}\left((an+b)^2\right) = \frac{z\mathbb{P}_{a,b,2}(z)}{(z-1)^3}. \tag{1.8}$$

When $a = 1, b = 0$ we obtain the known fact $\mathcal{Z}\left(n^2\right) = \frac{z\mathbb{P}_{1,0,2}(z)}{(z-1)^3}$, involving the Eulerian polynomial $\mathbb{P}_{1,0,2}(z) = \mathbb{P}_2(z) = z+1$ —see (1.5)—. The generalized Eulerian polynomials, like

(1.7), appear in [9]. However, this kind of polynomials is a particular case of the generalization we consider in [7]. We will suppose that $a \neq 0$ (otherwise there is nothing interesting to say: (1.8) reduces to $\mathcal{Z}(1) = \frac{z}{z-1}$).

Observe that

$$\mathbb{P}_{a,a-b,2}(z) = (a-b)^2 z^2 + \left(a^2 + 2ab - 2b^2\right) z + b^2, \tag{1.9}$$

is the reciprocal polynomial of $\mathbb{P}_{a,b,2}(z)$, that is, we have

$$\mathbb{P}_{a,a-b,2}(z) = z^2 \mathbb{P}_{a,b,2}\left(z^{-1}\right). \tag{1.10}$$

According to (1.3) and (1.8), we have

$$\begin{aligned} \mathcal{Z}\left((a(n+m)+b)^2\right) &= z^m \left(\mathcal{Z}\left((an+b)^2\right) - \sum_{j=0}^{m-1} \frac{(aj+b)^2}{z^j}\right) \qquad (1.11) \\ &= z^m \left(\frac{z\mathbb{P}_{a,b,2}(z)}{(z-1)^3} - \sum_{j=0}^{m-1} \frac{(aj+b)^2}{z^j}\right), \end{aligned}$$

where m is any non-negative integer. On the other hand, we have also from (1.8) that

$$\mathcal{Z}\left((a(n+m)+b)^2\right) = \mathcal{Z}\left((an+am+b)^2\right) = \frac{z\mathbb{P}_{a,am+b,2}(z)}{(z-1)^3}. \tag{1.12}$$

Thus, from (1.11) and (1.12) we see that

$$z^m \left(\frac{z\mathbb{P}_{a,b,2}(z)}{(z-1)^3} - \sum_{j=0}^{m-1} \frac{(aj+b)^2}{z^j}\right) = \frac{z\mathbb{P}_{a,am+b,2}(z)}{(z-1)^3}, \tag{1.13}$$

or

$$\sum_{j=0}^{m-1} \frac{(aj+b)^2}{z^j} = \frac{z\mathbb{P}_{a,b,2}(z)}{(z-1)^3} - z^{-m} \frac{z\mathbb{P}_{a,am+b,2}(z)}{(z-1)^3}. \tag{1.14}$$

Let us substitute z by z^{-1}, to obtain

$$\sum_{j=0}^{m-1} z^j (aj+b)^2 = \frac{z^2\mathbb{P}_{a,b,2}\left(z^{-1}\right)}{(1-z)^3} - z^m \frac{z^2\mathbb{P}_{a,am+b,2}\left(z^{-1}\right)}{(1-z)^3}, \tag{1.15}$$

which can be written in terms of the corresponding reciprocal generalized Eulerian polynomials (1.10) as

$$\sum_{j=0}^{m-1} z^j (aj+b)^2 = \frac{\mathbb{P}_{a,a-b,2}(z)}{(1-z)^3} - z^m \frac{\mathbb{P}_{a,a(1-m)-b,2}(z)}{(1-z)^3}. \tag{1.16}$$

This is an explicit formula for the weighted sum of squares $\sum_{j=0}^{m-1} z^j (aj+b)^2$ (where $z \neq 0, 1$ is the weight), in terms of the generalized Eulerian polynomials $\mathbb{P}_{a,a-b,2}(z)$, $\mathbb{P}_{a,a(1-m)-b,2}(z)$.

We comment in passing that there is a natural similar discussion for weighted sums of the p-th powers of the sequence $an+b$. It turns out that the Z-transform of the sequence $(an+b)^p$ is of the form $\mathcal{Z}((an+b)^p) = \frac{z\mathbb{P}_{a,b,p}(z)}{(z-1)^{p+1}}$, where $\mathbb{P}_{a,b,p}(z)$ is a p-th degree polynomial —the generalized Eulerian polynomial of p-th degree—, with recurrence

$$\mathbb{P}_{a,b,p}(z) = az(1-z)\mathbb{P}'_{a,b,p-1}(z) + (a-b+(a(p-1)+b)z)\mathbb{P}_{a,b,p-1}(z).$$

The corresponding expression for the weighted sum $\sum_{j=0}^{m-1} z^j (aj+b)^p$ —that generalizes (1.16)— is

$$\sum_{j=0}^{m-1} z^j (aj+b)^p = \frac{1}{(1-z)^{p+1}} \left(\mathbb{P}_{a,a-b,p}(z) - z^m \mathbb{P}_{a,a(1-m)-b,p}(z)\right). \tag{1.17}$$

(The case $a=1$ and $b=0$, is considered in [1].) It turns out that some problems related with (weighted) sums of powers are naturally related with Eulerian polynomials, or with some generalizations of them (see [5]).

In this work we are interested in some particular cases of formula (1.16). In Section 2 we will obtain explicit results of weighted sums (1.16), by considering some weights $z \in \mathbb{C}$ that produce 'simple' expressions for the corresponding right-hand side of (1.16) (the value of the weighted sum). In Section 3 we consider a particular case of the results of Section 2: the case of weighted sums with rational weights. We will see that Primitive Pythagorean Triples appear in a natural way in the sum terms, and in the value of the sum as well. Finally, in Section 4 we make some comments about a possible continuation of this work, in which the squares of the weighted sums are replaced by sums of two or three squares (we give some concrete examples). Some of the results in Section 3 were obtained in a recent work [6], by working with a different approach.

2. Weighted sums of squares I: general case

Let us begin by noting that the polynomial $\mathbb{P}_{a,a(1-m)-b,2}(z)$ (in (1.16)) can be written as

$$\mathbb{P}_{a,a(1-m)-b,2}(z) = am(1-z)((am+2b)(1-z)+2az) + \mathbb{P}_{a,a-b,2}(z).$$

Thus, we can write (1.16) as

$$\sum_{j=0}^{m-1} z^j (aj+b)^2 = \frac{1-z^m}{(1-z)^3}\mathbb{P}_{a,a-b,2}(z) - z^m \frac{am}{1-z}\left(am+2b+\frac{2az}{1-z}\right). \tag{2.1}$$

or (by changing b by $a-b$)

$$\sum_{j=0}^{m-1} z^j (a(j+1)-b)^2 = \frac{1-z^m}{(1-z)^3}\mathbb{P}_{a,b,2}(z) - z^m \frac{am}{1-z}\left(am-2b-\frac{2a}{z-1}\right). \tag{2.2}$$

We want to obtain explicit expressions of weighted sums of squares (2.2) (with concrete weights z and/or parameters a,b). Of course, we want "simple right-hand side" expressions, for example, with weights z such that $\frac{1-z^m}{(1-z)^3}\mathbb{P}_{a,b,2}(z) = 0$ or $z^m \frac{am}{1-z}\left(am-2b-\frac{2a}{z-1}\right) = 0$ (that disappear 'the half' of the right-hand side of (2.2)). It turns out that the big surprises occur with weights such that $\mathbb{P}_{a,b,2}(z) = 0$, and this will be the main discussion of this section. However, we want to give now some examples of the remaining cases.

If we take weights of the form $z_0 = \frac{2a+am-2b}{am-2b}$, which makes 0 the second term of the right-hand side of (2.2), we obtain

$$\sum_{j=0}^{m-1} z_0^j (a(j+1)-b)^2 = \frac{1-z_0^m}{(1-z_0)^3}\mathbb{P}_{a,b,2}(z_0). \tag{2.3}$$

A concrete example from (2.3) is the following

$$\sum_{j=0}^{m-1}\left(\frac{2+m}{m}\right)^{j}(j+1)^{2} \quad = \quad \frac{1}{4}m^{2}(m+1)\left(\left(\frac{m+2}{m}\right)^{m}-1\right).$$

For $s\in\mathbb{N}$ we can write (2.2) as

$$\sum_{j=0}^{sm-1} z^{j}\left(a(j+1)-b\right)^{2}=\frac{1-z^{sm}}{(1-z)^{3}}\mathbb{P}_{a,b,2}(z)-z^{sm}\frac{asm}{1-z}\left(asm-2b-\frac{2a}{z-1}\right). \tag{2.4}$$

If s is even, $2s$ say, we can take $z=-1$ (and then $1-z^{sm}=0$), to write (2.4) as

$$\sum_{j=0}^{2sm-1}(-1)^{j}\left(a(j+1)-b\right)^{2}=-asm\left(a(2sm+1)-2b\right), \tag{2.5}$$

where $s\in\mathbb{N}$. (Formulas for alternate sums of powers, like (2.5), are well-known results. See for example [2], formula 24.4.10, where the alternate sum of powers $\sum_{j=0}^{m-1}(-1)^{j}(aj+b)^{p}$ is expressed in terms of Euler polynomials.)

We consider now the most important case in this section, namely, when the weight $z_0\in\mathbb{C}$ is a root of $\mathbb{P}_{a,b,2}(z)=0$. In this case we can write (2.2) as

$$\sum_{j=0}^{m-1} z_0^{j-m}\left(a(j+1)-b\right)^{2}=\frac{am}{z_0-1}\left(am-2b-\frac{2a}{z_0-1}\right). \tag{2.6}$$

Observe that if $a=b$, the polynomial (1.7) is $\mathbb{P}_{a,a,2}(z)=a^{2}z(z+1)$. Thus we have only the weight $z_0=-1$ and (2.6) becomes $\sum_{j=0}^{m-1}(-1)^{j-m}j^{2}=-\binom{m}{2}$, which is a well-known result. Thus, we will suppose that $a\neq b$. Note that this fact ($a\neq b$) implies that $z_0=0$ is not a root of $\mathbb{P}_{a,b,2}(z)=0$. Moreover, $z_0=1$ is neither a root of $\mathbb{P}_{a,b,2}(z)=0$, since $\mathbb{P}_{a,b,2}(1)=2a^{2}$. Observe also that if $b=0$, the polynomial (1.7) is $\mathbb{P}_{a,0,2}(z)=a^{2}(z+1)$, which produces only the weight $z_0=-1$, and the corresponding weighted sum (2.6) is just a shifted version of above weighted sum $\sum_{j=0}^{m-1}(-1)^{j-m}j^{2}$. Thus, we will suppose also that $b\neq 0$. Summarizing: the roots of $\mathbb{P}_{a,b,2}(z)=0$ in the considered cases ($a\neq b$, $a\neq 0$, $b\neq 0$), are “good weights” for the weighted sum (2.6).

Since $\mathbb{P}_{a,a-b,2}(z)$ is the reciprocal polynomial of $\mathbb{P}_{a,b,2}(z)$ (and then $z_0^{-1}\in\mathbb{C}$ is a root of $\mathbb{P}_{a,a-b,2}(z)=0$), we have also (from (2.1)) the weighted sum

$$\sum_{j=0}^{m-1}\left(z_0^{-1}\right)^{j-m}(aj+b)^{2}=\frac{am}{z_0^{-1}-1}\left(am+2b+\frac{2az_0^{-1}}{1-z_0^{-1}}\right). \tag{2.7}$$

The roots z_0 of $\mathbb{P}_{a,b,2}(z)=0$ are

$$z_0=\frac{-\left(a^{2}+2ab-2b^{2}\right)\pm ar}{2b^{2}}, \tag{2.8}$$

where

$$r^{2}=a^{2}+4ab-4b^{2}. \tag{2.9}$$

Formula (2.9) implies the following algebraic relations (we leave the verifications to the reader)

$$
\begin{aligned}
(a+2b-r)^2 &= -2\left(ar-4ab+2br-a^2\right), \\
\frac{2b\,(a-r)}{a-r+2b} &= -\frac{1}{2}\left(a+r-2b\right), \\
a-\left(-b\frac{a-r-2b}{a-r+2b}\right) &= -b\frac{a+r-2b}{a+r+2b}, \\
(a+r)^2+4b^2 &= 2a\left(a+r+2b\right).
\end{aligned}
$$

These formulas remain valid if we replace r by $-r$, and we will be using them without further comments. It turns out that they are useful formulas to deal with the algebraic manipulations involved in the task we begin now: to obtain explicit expressions of the weighted sums (2.1) and (2.2) in terms of the parameters a, b and r.

We can write the two roots (2.8) together as

$$z_0 = -\left(\frac{a\pm r}{2b}\right)^2. \tag{2.10}$$

The corresponding weighted sums (2.2) are

$$\sum_{j=0}^{m-1}\left(-\left(\frac{a\pm r}{2b}\right)^2\right)^{j-m}(aj+a-b)^2 = -\frac{2b^2 m}{a\pm r+2b}\left(am+\frac{1}{2}\left(a\mp r-2b\right)\right), \tag{2.11}$$

and the corresponding weighted sums (2.1) are

$$\sum_{j=0}^{m-1}\left(-\left(\frac{2b}{a\pm r}\right)^2\right)^{j-m}(aj+b)^2 = -\frac{(a\pm r)^2\,m}{2\left(a\pm r+2b\right)}\left(am-\frac{1}{2}\left(a\mp r-2b\right)\right),$$

that we can write as

$$\sum_{j=0}^{m-1}\left(-\left(\frac{2b}{a\pm r}\right)^2\right)^{j-m+1}(aj+b)^2 = \frac{2b^2 m}{a\pm r+2b}\left(am-\frac{1}{2}\left(a\mp r-2b\right)\right). \tag{2.12}$$

Then we have, so far, four weighted sums (2.11) and (2.12). But the story continues: if we replace b by $-b\frac{a-r-2b}{a-r+2b}$ in (1.7), after some algebraic work with the corresponding polynomial $\mathbb{P}_{a,-b\frac{a-r-2b}{a-r+2b},2}(z)$, we can write the equation $\mathbb{P}_{a,-b\frac{a-r-2b}{a-r+2b},2}(z)=0$ as

$$(a+r)^2 z^2 + 4\left(a^2-2ab+2b^2\right)z + (a-r)^2 = 0. \tag{2.13}$$

The roots of this equation are

$$z_0 = -\left(\frac{2b}{a+r}\right)^2, -\left(\frac{a-r}{2b}\right)^2, \tag{2.14}$$

and the weighted sum (2.6) becomes

$$\sum_{j=0}^{m-1} z_0^{j-m}\left(aj+a+b\frac{a-r-2b}{a-r+2b}\right)^2 = \frac{am}{z_0-1}\left(am-\frac{2a}{z_0-1}+2b\frac{a-r-2b}{a-r+2b}\right). \tag{2.15}$$

With $z_0 = -\left(\frac{2b}{a+r}\right)^2$, the weighted sum (2.15) is

$$\sum_{j=0}^{m-1}\left(-\left(\frac{2b}{a+r}\right)^2\right)^{j-m}\left(aj+a+b\frac{a-r-2b}{a-r+2b}\right)^2 = -\frac{1}{2}\frac{(a+r)^2 m}{a+r+2b}\left(am+\frac{1}{2}(a-r-2b)\right). \tag{2.16}$$

With $z_0 = -\left(\frac{a-r}{2b}\right)^2$, the weighted sum (2.15) is

$$\sum_{j=0}^{m-1}\left(-\left(\frac{a-r}{2b}\right)^2\right)^{j-m}\left(aj+a+b\frac{a-r-2b}{a-r+2b}\right)^2 = -\frac{2b^2 m}{a-r+2b}\left(am-\frac{1}{2}(a+r-2b)\right). \tag{2.17}$$

The reciprocal equation of $\mathbb{P}_{a,-b\frac{a-r-2b}{a-r+2b},2}(z) = 0$ comes by substituting $-b\frac{a-r-2b}{a-r+2b}$ by $a - \left(-b\frac{a-r-2b}{a-r+2b}\right) = -b\frac{a+r-2b}{a+r+2b}$. That is, the reciprocal equation of $\mathbb{P}_{a,-b\frac{a-r-2b}{a-r+2b},2}(z) = 0$ is the equation $\mathbb{P}_{a,-b\frac{a+r-2b}{a+r+2b},2}(z) = 0$ (obtained by changing r by $-r$ in $\mathbb{P}_{a,-b\frac{a-r-2b}{a-r+2b},2}(z) = 0$, as it is evident from (2.13)). The roots of the reciprocal equation $\mathbb{P}_{a,-b\frac{a+r-2b}{a+r+2b},2}(z) = 0$ are plainly $z_0^{-1} = -\left(\frac{a+r}{2b}\right)^2, -\left(\frac{2b}{a-r}\right)^2$, and the corresponding weighted sums

$$\sum_{j=0}^{m-1}\left(z_0^{-1}\right)^{j-m}\left(aj+a+b\frac{a+r-2b}{a+r+2b}\right)^2 = \frac{am}{z_0^{-1}-1}\left(am-\frac{2a}{z_0^{-1}-1}+2b\frac{a+r-2b}{a+r+2b}\right), \tag{2.18}$$

are as follows: for $z_0^{-1} = -\left(\frac{a+r}{2b}\right)^2$, we have the weighted sum

$$\sum_{j=0}^{m-1}\left(-\left(\frac{a+r}{2b}\right)^2\right)^{j-m}\left(aj+a+b\frac{a+r-2b}{a+r+2b}\right)^2 = -\frac{2b^2 m}{a+r+2b}\left(am-\frac{1}{2}(a-r-2b)\right). \tag{2.19}$$

For $z_0^{-1} = -\left(\frac{2b}{a-r}\right)^2$, we have the weighted sum

$$\sum_{j=0}^{m-1}\left(-\left(\frac{2b}{a-r}\right)^2\right)^{j-m}\left(aj+a+b\frac{a+r-2b}{a+r+2b}\right)^2 = -\frac{(a-r)^2 m}{2(a-r+2b)}\left(am+\frac{1}{2}(a+r-2b)\right). \tag{2.20}$$

We have now four more weighted sums, namely (2.16), (2.17), (2.19) and (2.20).

We can write together (2.16) and (2.20) as

$$\sum_{j=0}^{m-1}\left(-\left(\frac{2b}{a\pm r}\right)^2\right)^{j-m}\left(aj+a+b\frac{a\mp r-2b}{a\mp r+2b}\right)^2 = -\frac{1}{2}\frac{(a\pm r)^2 m}{a\pm r+2b}\left(am+\frac{1}{2}(a\mp r-2b)\right),$$

or

$$\sum_{j=0}^{m-1}\left(-\left(\frac{2b}{a\pm r}\right)^2\right)^{j-m+1}\left(aj+a+b\frac{a\mp r-2b}{a\mp r+2b}\right)^2 = \frac{2b^2 m}{a\pm r+2b}\left(am+\frac{1}{2}(a\mp r-2b)\right). \tag{2.21}$$

Similarly, (2.17) and (2.19) can be written together as

$$\sum_{j=0}^{m-1}\left(-\left(\frac{a\pm r}{2b}\right)^2\right)^{j-m}\left(aj+a+b\frac{a\pm r-2b}{a\pm r+2b}\right)^2=-\frac{2b^2m}{a\pm r+2b}\left(am-\frac{1}{2}\left(a\mp r-2b\right)\right). \tag{2.22}$$

Summarizing, we have in total eight weighted sums of squares: two in each of the formulas (2.11), (2.12), (2.21) and (2.22).

Let us make some additional remarks: if we replace m by $-m$ in (2.12) and (2.22), we obtain

$$\sum_{j=0}^{-m-1}\left(-\left(\frac{2b}{a\pm r}\right)^2\right)^{j+m+1}(aj+b)^2=\frac{2b^2m}{a\pm r+2b}\left(am+\frac{1}{2}\left(a\mp r-2b\right)\right), \tag{2.23}$$

and

$$\sum_{j=0}^{-m-1}\left(-\left(\frac{a\pm r}{2b}\right)^2\right)^{j+m}\left(aj+a+b\frac{a\pm r-2b}{a\pm r+2b}\right)^2=-\frac{2b^2m}{a\pm r+2b}\left(am+\frac{1}{2}\left(a\mp r-2b\right)\right), \tag{2.24}$$

respectively.

(For sums of the form $\sum_{k=0}^{M}\varphi(k)$, where $-M\in\mathbb{N}$, we use the convention $\sum_{k=0}^{-n-1}\varphi(k)=-\sum_{k=-n}^{-1}\varphi(k)$, where $n\in\mathbb{N}$.)

The expressions (2.11), (2.21), (2.23) and (2.24), show us that the eight weighted sums involved in them, can be written —according to their right-hand sides—, in only *two kinds of weighted sums of squares.* These are the main results of this work, and we write them in the following proposition.

Proposition 2.1. *For complex parameters a,b ($a\neq 0, b\neq 0, a\neq b$), and $r=\sqrt{a^2+4ab-4b^2}$, we have the following weighted sums of squares*

$$\begin{aligned}\sum_{j=0}^{m-1}\left(-\left(\frac{a+r}{2b}\right)^2\right)^{j-m}(aj+a-b)^2 &= -\sum_{j=0}^{-m-1}\left(-\left(\frac{2b}{a+r}\right)^2\right)^{j+m+1}(aj+b)^2\\ &= -\sum_{j=0}^{m-1}\left(-\left(\frac{2b}{a+r}\right)^2\right)^{j-m+1}\left(aj+a+b\frac{a-r-2b}{a-r+2b}\right)^2\\ &= \sum_{j=0}^{-m-1}\left(-\left(\frac{a+r}{2b}\right)^2\right)^{j+m}\left(aj+a+b\frac{a+r-2b}{a+r+2b}\right)^2\\ &= -\frac{2b^2m}{a+r+2b}\left(am+\frac{1}{2}\left(a-r-2b\right)\right),\end{aligned} \tag{2.25}$$

and

$$\begin{aligned}
\sum_{j=0}^{m-1}\left(-\left(\frac{a-r}{2b}\right)^2\right)^{j-m}(aj+a-b)^2 &= -\sum_{j=0}^{-m-1}\left(-\left(\frac{2b}{a-r}\right)^2\right)^{j+m+1}(aj+b)^2\\
&= -\sum_{j=0}^{m-1}\left(-\left(\frac{2b}{a-r}\right)^2\right)^{j-m+1}\left(aj+a+b\frac{a+r-2b}{a+r+2b}\right)^2\\
&= \sum_{j=0}^{-m-1}\left(-\left(\frac{a-r}{2b}\right)^2\right)^{j+m}\left(aj+a+b\frac{a-r-2b}{a-r+2b}\right)^2\\
&= -\frac{2b^2m}{a-r+2b}\left(am+\frac{1}{2}\left(a+r-2b\right)\right). \qquad (2.26)
\end{aligned}$$

Observe that the weights are of the form $-\lambda_1^2$, $-\lambda_2^2$, $-\left(\lambda_1^{-1}\right)^2$, $-\left(\lambda_2^{-1}\right)^2$, where

$$\lambda_1=\frac{a+r}{2b} \quad \text{and} \quad \lambda_2=\frac{a-r}{2b}. \qquad (2.27)$$

Observe also that there are (only) four “independent terms”, which appear in the squares involved in the weighted sums (2.25) and (2.26), namely

$$\beta_1=a-b \ \ , \ \ \beta_2=b \ \ , \ \ \beta_3=a+b\frac{a+r-2b}{a+r+2b} \quad \text{and} \quad \beta_4=a+b\frac{a-r-2b}{a-r+2b}. \qquad (2.28)$$

More precisely, β_1 (β_3) appears in the weighted sums with positive m and weights $-\lambda_1^2$ and $-\lambda_2^2$ ($-\left(\lambda_1^{-1}\right)^2$ and $-\left(\lambda_2^{-1}\right)^2$, respectively); β_2 (β_4) appears in the weighted sums with negative m and weights $-\left(\lambda_1^{-1}\right)^2$ and $-\left(\lambda_2^{-1}\right)^2$ ($-\lambda_1^2$ and $-\lambda_2^2$, respectively).

One more observation for the weighted sums (2.25) and (2.26) is that they are determined by the values of the parameters a and b (and r, according to (2.9)), and that multiples of these parameters, na and nb (and then also nr), say, with $n \neq 0$, produce the same weighted sums (2.25) and (2.26).

There is a nice formula involving the “independent terms” of the parentheses in the right-hand sides of (2.25) and (2.26), namely $\frac{1}{2}\left(a-r-2b\right)$ and $\frac{1}{2}\left(a+r-2b\right)$, respectively. The formula is

$$\left(\frac{1}{2}\left(a+r-2b\right)\right)^2+\left(\frac{1}{2}\left(a-r-2b\right)\right)^2=a^2. \qquad (2.29)$$

(The verification is straightforward.) There are some interesting consequences of (2.29), related (of course) to the Pythagorean Theorem, that we will discuss in the next section.

An example of the weighted sums of squares (2.25) and (2.26), for a real sequence of squares with complex weights (corresponding to $a = 4, b = -1, r = 2i$), is the following:

$$\begin{aligned}
\sum_{k=0}^{m-1} (-3 \mp 4i)^{k-m} (4k+5)^2 &= -\sum_{k=0}^{-m-1} \left(-\frac{3}{25} \pm \frac{4}{25} i\right)^{k+m+1} (4k-1)^2 \\
&= -\sum_{k=0}^{m-1} \left(-\frac{3}{25} \pm \frac{4}{25} i\right)^{k-m+1} (4k+2 \mp i)^2 \\
&= \sum_{k=0}^{-m-1} (-3 \mp 4i)^{k+m} (4k+2 \pm i)^2 \\
&= -\frac{1}{2} (1 \mp i)\, m\, (4m+3 \mp i).
\end{aligned}$$

Formula (2.29) looks in this example as $(3+i)^2 + (3-i)^2 = 16 = a^2$.

3. Weighted sums of squares II: rational weights

In this section we will consider the weighted sums (2.25) and (2.26) where the parameters a, b are *integers* ($a \neq 0, b \neq 0, a \neq b$), and the weights (2.27) are *rational*, that is, the integer $a^2 + 4ab - 4b^2$ is a square. We continue writing r^2 for this number (see (2.9), where now $r \in \mathbb{Z}$). Observe that in this case, the numbers $a \pm r$ are even. In fact, $(a \pm r)^2 = 2a^2 + 4ab - 4b^2 \pm 2ar$, which shows that $(a \pm r)^2$ are even, and then $a \pm r$ are even. Formula (2.29) tells us that $\left(\frac{1}{2}|a + r - 2b|, \frac{1}{2}|a - r - 2b|, |a|\right)$ is a Pythagorean triple. Moreover, if we have any Pythagorean triple $(C_1, C_2, |a|)$, we can solve the system $\frac{1}{2}|a + r - 2b| = C_1$, $\frac{1}{2}|a - r - 2b| = C_2$, for b and r, to obtain

$$\begin{aligned}
b &= \frac{1}{2}(a \mp C_1 \mp C_2), \\
r &= \pm C_1 \mp C_2.
\end{aligned} \tag{3.1}$$

We can restrict ourselves to Primitive Pythagorean Triples (PPT, for short). (See the observations after Proposition 2.1.)

We have four possibilities for the pair (b, r), namely

$$\begin{aligned}
(b,r)_1 &= \left(\tfrac{1}{2}(a - C_1 - C_2), C_1 - C_2\right), & (b,r)_2 &= \left(\tfrac{1}{2}(a - C_1 + C_2), C_1 + C_2\right), \\
(b,r)_3 &= \left(\tfrac{1}{2}(a + C_1 - C_2), -C_1 - C_2\right), & (b,r)_4 &= \left(\tfrac{1}{2}(a + C_1 + C_2), -C_1 + C_2\right).
\end{aligned} \tag{3.2}$$

We consider only the case $a > 0$, since for $a < 0$, the corresponding values for the pair (b, r) are the negatives of the values (3.2). Thus, for each PPT, it seems that we can have four sets of weighted sums (2.25) and (2.26) (32 in total!). For example, for the simplest PPT $(3, 4, 5)$, we have the possibilities $(b, r)_1 = (-1, -1)$, $(b, r)_2 = (3, 7)$, $(b, r)_3 = (2, -7)$ and $(b, r)_4 = (6, 1)$. (In the case $a = -5$, we have $(b, r)_5 = (-6, -1) = -(b, r)_4$, $(b, r)_6 = (-2, 7) = -(b, r)_3$, $(b, r)_7 = (-3, -7) = -(b, r)_2$ and $(b, r)_8 = (1, 1) = -(b, r)_1$, so there are not new weighted sums if $a = -5$.)

For each of the values (3.2), we can translate the work we did in Section 2 to the language of PPT's. For example, let us consider the first pair of the list (3.2):

$$(b, r)_1 = \left(\frac{1}{2}(a - C_1 - C_2), C_1 - C_2\right).$$

The PPT version of the numbers λ_1 and λ_2 involved in the weights (see (2.27)) corresponding to $(b,r)_1$ is

$$\lambda_1 = \frac{a+C_1-C_2}{a-C_1-C_2} \quad \text{and} \quad \lambda_2 = \frac{a-C_1+C_2}{a-C_1-C_2}. \tag{3.3}$$

The PPT version of the "independent terms" (2.28) corresponding to $(b,r)_1$ is

$$\begin{aligned} \beta_1 &= \frac{1}{2}(a+C_1+C_2)\ ,\ \beta_2 = \frac{1}{2}(a-C_1-C_2)\,, \\ \beta_3 &= \frac{1}{2}(a-C_1+C_2)\ ,\ \beta_4 = \frac{1}{2}(a+C_1-C_2)\,. \end{aligned} \tag{3.4}$$

The PPT version of the weighted sums (2.25) and (2.26) corresponding to $(b,r)_1$ is

$$\begin{aligned} & \sum_{j=0}^{m-1}\left(-\left(\frac{a+C_1-C_2}{a-C_1-C_2}\right)^2\right)^{j-m}\left(aj+\frac{1}{2}(a+C_1+C_2)\right)^2 \\ = & -\sum_{j=0}^{-m-1}\left(-\left(\frac{a-C_1-C_2}{a+C_1-C_2}\right)^2\right)^{j+m+1}\left(aj+\frac{1}{2}(a-C_1-C_2)\right)^2 \\ = & -\sum_{j=0}^{m-1}\left(-\left(\frac{a-C_1-C_2}{a+C_1-C_2}\right)^2\right)^{j-m+1}\left(aj+\frac{1}{2}(a-C_1+C_2)\right)^2 \\ = & \sum_{j=0}^{-m-1}\left(-\left(\frac{a+C_1-C_2}{a-C_1-C_2}\right)^2\right)^{j+m}\left(aj+\frac{1}{2}(a+C_1-C_2)\right)^2 \\ = & -\frac{1}{2}(a-C_1)\,m\,(am+C_2)\,, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} & \sum_{j=0}^{m-1}\left(-\left(\frac{a-C_1+C_2}{a-C_1-C_2}\right)^2\right)^{j-m}\left(aj+\frac{1}{2}(a+C_1+C_2)\right)^2 \\ = & -\sum_{j=0}^{-m-1}\left(-\left(\frac{a-C_1-C_2}{a-C_1+C_2}\right)^2\right)^{j+m+1}\left(aj+\frac{1}{2}(a-C_1-C_2)\right)^2 \\ = & -\sum_{j=0}^{m-1}\left(-\left(\frac{a-C_1-C_2}{a-C_1+C_2}\right)^2\right)^{j-m+1}\left(aj+\frac{1}{2}(a+C_1-C_2)\right)^2 \\ = & \sum_{j=0}^{-m-1}\left(-\left(\frac{a-C_1+C_2}{a-C_1-C_2}\right)^2\right)^{j+m}\left(aj+\frac{1}{2}(a-C_1+C_2)\right)^2 \\ = & -\frac{1}{2}(a-C_2)\,m\,(am+C_1)\,. \end{aligned} \tag{3.6}$$

Even though formulas (3.5) and (3.6) have an 'attracting beauty', we prefer the following simpler version of them (involving (3.3) and (3.4)):

$$\begin{aligned}
&\sum_{j=0}^{m-1}\left(-\lambda_1^2\right)^{j-m}(aj+\beta_1)^2 = -\sum_{j=0}^{-m-1}\left(-\frac{1}{\lambda_1^2}\right)^{j+m+1}(aj+\beta_2)^2 \qquad (3.7)\\
= \;& -\sum_{j=0}^{m-1}\left(-\frac{1}{\lambda_1^2}\right)^{j-m+1}(aj+\beta_3)^2 = \sum_{j=0}^{-m-1}\left(-\lambda_1^2\right)^{j+m}(aj+\beta_4)^2\\
= \;& -\frac{1}{2}(a-C_1)\,m\,(am+C_2),
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{j=0}^{m-1}\left(-\lambda_2^2\right)^{j-m}(aj+\beta_1)^2 = -\sum_{j=0}^{-m-1}\left(-\frac{1}{\lambda_2^2}\right)^{j+m+1}(aj+\beta_2)^2 \qquad (3.8)\\
= \;& -\sum_{j=0}^{m-1}\left(-\frac{1}{\lambda_2^2}\right)^{j-m+1}(aj+\beta_4)^2 = \sum_{j=0}^{-m-1}\left(-\lambda_2^2\right)^{j+m}(aj+\beta_3)^2\\
= \;& -\frac{1}{2}(a-C_2)\,m\,(am+C_1),
\end{aligned}$$

respectively. (Moreover, we can have a 'β version' of these formulas, noting that $\lambda_1 = \frac{\beta_4}{\beta_2}$ and $\lambda_2 = \frac{\beta_3}{\beta_2}$.)

Surprisingly, the corresponding weighted sums for each of the remaining values $(b,r)_2$, $(b,r)_3$ and $(b,r)_4$ in (3.2) are *the same* as (3.7) and (3.8) (obtained with the value $(b,r)_1$). We will shortly prove this affirmation. The following formulas will be helpful in the algebraic manipulations we will face in the proof:

$$(\lambda_1 =)\,\frac{a+C_1-C_2}{a-C_1-C_2} = -\frac{a+C_1+C_2}{a-C_1+C_2}, \qquad (3.9)$$

$$(\lambda_2 =)\,\frac{a-C_1+C_2}{a-C_1-C_2} = -\frac{a+C_1+C_2}{a+C_1-C_2}, \qquad (3.10)$$

$$\lambda_1^2 = \frac{a+C_1}{a-C_1}, \qquad (3.11)$$

$$\lambda_2^2 = \frac{a+C_2}{a-C_2}. \qquad (3.12)$$

Let us verify (3.9). This formula is equivalent to

$$(a+C_1-C_2)(a-C_1+C_2)+(a+C_1+C_2)(a-C_1-C_2)=0,$$

which in turn is equivalent to $a^2-C_1^2-C_2^2=0$, which is obvious (since (C_1,C_2,a) is a Pythagorean triple). The verification of (3.10) is similar. Let us verify (3.11). We have

$$\lambda_1^2 = \frac{(a+C_1-C_2)^2}{(a-C_1-C_2)^2} = \frac{2a^2+2aC_1-2aC_2-2C_1C_2}{2a^2-2aC_1-2aC_2+2C_1C_2} = \frac{a+C_1}{a-C_1}.$$

The verification of (3.12) is similar.

Lemma 3.1. *Let (C_1,C_2,a) be a PPT. The weighted sums obtained from (2.25) and (2.26), substituting in them the values $(b,r)_2$, $(b,r)_3$ or $(b,r)_4$ (from (3.2)), are equal to the weighted* *sums (3.7) and (3.8), respectively (which were obtained from (2.25) and (2.26), with the value* $(b,r)_1$*).*

Proof. With $(b,r)_2 = \left(\frac{1}{2}(a - C_1 + C_2), C_1 + C_2\right)$, the weighted sums (2.25) and (2.26) are, respectively

$$\begin{aligned}
&\sum_{j=0}^{m-1}\left(-\left(\frac{a+C_1+C_2}{a-C_1+C_2}\right)^2\right)^{j-m}\left(aj+\frac{1}{2}(a+C_1-C_2)\right)^2 \qquad (3.13)\\
=\ & -\sum_{j=0}^{-m-1}\left(-\left(\frac{a-C_1+C_2}{a+C_1+C_2}\right)^2\right)^{j+m+1}\left(aj+\frac{1}{2}(a-C_1+C_2)\right)^2\\
=\ & -\sum_{j=0}^{m-1}\left(-\left(\frac{a-C_1+C_2}{a+C_1+C_2}\right)^2\right)^{j-m+1}\left(aj+\frac{1}{2}(a-C_1-C_2)\right)^2\\
=\ & \sum_{j=0}^{-m-1}\left(-\left(\frac{a+C_1+C_2}{a-C_1+C_2}\right)^2\right)^{j+m}\left(aj+\frac{1}{2}(a+C_1+C_2)\right)^2\\
=\ & -\frac{1}{2}(a-C_1)\,m\,(am-C_2),
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{j=0}^{m-1}\left(-\left(\frac{a-C_1-C_2}{a-C_1+C_2}\right)^2\right)^{j-m}\left(aj+\frac{1}{2}(a+C_1-C_2)\right)^2 \qquad (3.14)\\
=\ & -\sum_{j=0}^{-m-1}\left(-\left(\frac{a-C_1+C_2}{a-C_1-C_2}\right)^2\right)^{j+m+1}\left(aj+\frac{1}{2}(a-C_1+C_2)\right)^2\\
=\ & -\sum_{j=0}^{m-1}\left(-\left(\frac{a-C_1+C_2}{a-C_1-C_2}\right)^2\right)^{j-m+1}\left(aj+\frac{1}{2}(a+C_1+C_2)\right)^2\\
=\ & \sum_{j=0}^{-m-1}\left(-\left(\frac{a-C_1-C_2}{a-C_1+C_2}\right)^2\right)^{j+m}\left(aj+\frac{1}{2}(a-C_1-C_2)\right)^2\\
=\ & -\frac{1}{2}(a+C_2)\,m\,(am+C_1),
\end{aligned}$$

which can be written, by using (3.4), (3.9) and (3.10), as

$$\begin{aligned}
&\sum_{j=0}^{m-1}\left(-\lambda_1^2\right)^{j-m}(aj+\beta_4)^2\\
=\ & -\sum_{j=0}^{-m-1}\left(-\frac{1}{\lambda_1^2}\right)^{j+m+1}(aj+\beta_3)^2 = -\sum_{j=0}^{m-1}\left(-\frac{1}{\lambda_1^2}\right)^{j-m+1}(aj+\beta_2)^2\\
=\ & \sum_{j=0}^{-m-1}\left(-\lambda_1^2\right)^{j+m}(aj+\beta_1)^2 = -\frac{1}{2}(a-C_1)\,m\,(am-C_2), \qquad (3.15)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{j=0}^{m-1}\left(-\frac{1}{\lambda_2^2}\right)^{j-m}(aj+\beta_4)^2\\
&= -\sum_{j=0}^{-m-1}\left(-\lambda_2^2\right)^{j+m+1}(aj+\beta_3)^2 = -\sum_{j=0}^{m-1}\left(-\lambda_2^2\right)^{j-m+1}(aj+\beta_1)^2\\
&= \sum_{j=0}^{-m-1}\left(-\frac{1}{\lambda_2^2}\right)^{j+m}(aj+\beta_2)^2 = -\frac{1}{2}\left(a+C_2\right)m\left(am+C_1\right), \qquad (3.16)
\end{aligned}$$

respectively. Change m by $-m$ in (3.15) to obtain (3.7), and divide (3.16) by λ_2^2 (and use (3.12)), to obtain (3.8).

In the case of $(b,r)_3 = \left(\frac{1}{2}\left(a+C_1-C_2\right), -C_1-C_2\right)$, the weighted sums (2.25) and (2.26) can be be written, by using (3.4), (3.9) and (3.10), as

$$\begin{aligned}
&\sum_{j=0}^{m-1}\left(-\frac{1}{\lambda_1^2}\right)^{j-m}(aj+\beta_3)^2\\
&= -\sum_{j=0}^{-m-1}\left(-\lambda_1^2\right)^{j+m+1}(aj+\beta_4)^2 = -\sum_{j=0}^{m-1}\left(-\lambda_1^2\right)^{j-m+1}(aj+\beta_1)^2\\
&= \sum_{j=0}^{-m-1}\left(-\frac{1}{\lambda_1^2}\right)^{j+m}(aj+\beta_2)^2 = -\frac{1}{2}m\left(a+C_1\right)\left(am+C_2\right), \qquad (3.17)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{j=0}^{m-1}\left(-\lambda_2^2\right)^{j-m}(aj+\beta_3)^2\\
&= -\sum_{j=0}^{-m-1}\left(-\frac{1}{\lambda_2^2}\right)^{j+m+1}(aj+\beta_4)^2 = -\sum_{j=0}^{m-1}\left(-\frac{1}{\lambda_2^2}\right)^{j-m+1}(aj+\beta_2)^2\\
&= \sum_{j=0}^{-m-1}\left(-\lambda_2^2\right)^{j+m}(aj+\beta_1)^2 = -\frac{1}{2}m\left(a-C_2\right)\left(am-C_1\right), \qquad (3.18)
\end{aligned}$$

respectively. Divide (3.17) by λ_1^2 and use (3.11) to obtain (3.7). In (3.18) just change m by $-m$, to obtain (3.8).

In the case $(b,r)_4 = \left(\frac{1}{2}\left(a+C_1+C_2\right), -C_1+C_2\right)$, the weighted sums (2.25) and (2.26) can be written, by using (3.4), (3.9) and (3.10), as

$$\begin{aligned}
&\sum_{j=0}^{m-1}\left(-\frac{1}{\lambda_1^2}\right)^{j-m}(aj+\beta_2)^2\\
&= -\sum_{j=0}^{-m-1}\left(-\lambda_1^2\right)^{j+m+1}(aj+\beta_1)^2 = -\sum_{j=0}^{m-1}\left(-\lambda_1^2\right)^{j-m+1}(aj+\beta_4)^2\\
&= \sum_{j=0}^{-m-1}\left(-\frac{1}{\lambda_1^2}\right)^{j+m}(aj+\beta_3)^2 = -\frac{1}{2}\left(a+C_1\right)m\left(am-C_2\right), \qquad (3.19)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{j=0}^{m-1}\left(-\frac{1}{\lambda_2^2}\right)^{j-m}(aj+\beta_2)^2\\
=\;& -\sum_{j=0}^{-m-1}\left(-\lambda_2^2\right)^{j+m+1}(aj+\beta_1)^2=-\sum_{j=0}^{m-1}\left(-\lambda_2^2\right)^{j-m+1}(aj+\beta_3)^2\\
=\;& \sum_{j=0}^{-m-1}\left(-\frac{1}{\lambda_2^2}\right)^{j+m}(aj+\beta_4)^2=-\frac{1}{2}(a+C_2)\,m\,(am-C_1)\,, \qquad (3.20)
\end{aligned}$$

respectively. In (3.19) change m by $-m$, then divide by λ_1^2, and finally use (3.11) to obtain (3.7). In (3.20) change m by $-m$, then divide by λ_2^2, and finally use (3.12) to obtain (3.8). This ends the proof. □

Thus we have the following proposition (which proof comes directly from the previous discussion in this section) .

Proposition 3.2. *If $a,b\in\mathbb{Z}$ are such that $r^2=a^2+4ab-4b^2$ is a square, then the triple of integers $\left(\frac{1}{2}|a+r-2b|\,,\frac{1}{2}|a-r-2b|\,,|a|\right)$ is a Pythagorean triple that appears in the rational-weighted sums (2.25) and (2.26). Conversely, each PPT (C_1,C_2,a) produces a pair of weighted sums (2.25) and (2.26), where $(b,r)=\left(\frac{1}{2}(a-C_1-C_2)\,,C_1-C_2\right)$ (that is, produces the weighted sums (3.5) and (3.6), respectively).*

For example, let us consider the PPT $(3,4,5)$, and the pair $(b,r)_1=(-1,-1)$. The weighted sums (3.7) and (3.8) are

$$\begin{aligned}
\sum_{j=0}^{m-1}\left(-2^2\right)^{j-m}(5j+6)^2 &= -\sum_{j=0}^{-m-1}\left(-\frac{1}{2^2}\right)^{j+m+1}(5j-1)^2=-\sum_{j=0}^{m-1}\left(-\frac{1}{2^2}\right)^{j-m+1}(5j+3)^2\\
&= \sum_{j=0}^{-m-1}\left(-2^2\right)^{j+m}(5j+2)^2=-m\,(5m+4)\,,
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j=0}^{m-1}\left(-3^2\right)^{j-m}(5j+6)^2 &= -\sum_{j=0}^{-m-1}\left(-\frac{1}{3^2}\right)^{j+m+1}(5j-1)^2=-\sum_{j=0}^{m-1}\left(-\frac{1}{3^2}\right)^{j-m+1}(5j+2)^2\\
&= \sum_{j=0}^{-m-1}\left(-3^2\right)^{j+m}(5j+3)^2=-\frac{1}{2}m\,(5m+3)\,.
\end{aligned}$$

4. Further remarks

By using elementary linearity arguments, one can see that formula (1.16) can be written more generally as

$$\sum_{j=0}^{m-1}z^j\sum_{t=1}^{k}(a_tj+b_t)^2=\frac{1}{(1-z)^3}\sum_{t=1}^{k}\mathbb{P}_{a_t,a_t-b_t,2}(z)-\frac{z^m}{(1-z)^3}\sum_{t=1}^{k}\mathbb{P}_{a_t,a_t(1-m)-b_t,2}(z)\,, \qquad (4.1)$$

where $a_t, b_t \in \mathbb{C}$, $t = 1, \dots, k$ are given. This is a formula for the "weighted sum of sums of k squares" of the left-hand side. The solutions $z_0 \neq 0, 1$ of the (quadratic) equation

$$\sum_{t=1}^{k} \mathbb{P}_{a_t, a_t - b_t, 2}(z) = 0, \tag{4.2}$$

give us 'good weights', for which (4.1) has the simpler form

$$\sum_{j=0}^{m-1} z_0^j \sum_{t=1}^{k} (a_t j + b_t)^2 = -\frac{z_0^m}{(1-z_0)^3} \sum_{t=1}^{k} \mathbb{P}_{a_t, a_t(1-m)-b_t, 2}(z_0). \tag{4.3}$$

In the case $k = 1$, by seeking rational weights, we found the Primitive Pythagorean Triples involved in (4.3). What about $k = 2$ or $k = 3$? Considering rational solutions of (4.2), are there some special numbers involved in these cases? These questions provide a direction to continue this work.

We finish the article giving concrete examples of (4.1) in the cases $k = 2$ and $k = 3$.

- Weighted sums of sums of two squares:

$$\begin{aligned}
\sum_{j=0}^{m-1} \left(-\frac{\alpha^2}{\beta^2}\right)^{j-m} \left(((\alpha+\beta)j+\alpha)^2 + ((\alpha-\beta)j+\alpha)^2\right) &= -2\beta^2 m^2, \\
\sum_{j=0}^{m-1} (-1)^{j+m} \left(((\alpha+\beta)j+\alpha)^2 + ((\alpha-\beta)j+\alpha)^2\right) &= -m\left(\alpha^2(m+1) + \beta^2(m-1)\right).
\end{aligned}$$

- Weighted sums of sums of three squares:

$$\begin{aligned}
&\sum_{j=0}^{m-1} (-1)^{j-m} \left((3\alpha j+\alpha)^2 + (\beta j+\alpha)^2 + (\beta j-\alpha)^2\right) \\
= \ &\sum_{j=0}^{-m-1} (-1)^{j-m} \left((3\alpha j+2\alpha)^2 + (\beta j+\beta-\alpha)^2 + (\beta j+\beta+\alpha)^2\right) \\
= \ &-\frac{1}{2} m \left(2\beta^2(m-1) + 3\alpha^2(3m-1)\right), \\
&\sum_{j=0}^{m-1} \left(-\frac{2\left(\beta^2+3\alpha^2\right)}{3\alpha^2}\right)^{j-m} \left((3\alpha j+2\alpha)^2 + (\beta j+\beta-\alpha)^2 + (\beta j+\beta+\alpha)^2\right) \\
= \ &-\sum_{j=0}^{m-1} \left(-\frac{3\alpha^2}{2\left(\beta^2+3\alpha^2\right)}\right)^{j-m-1} \left((3\alpha j+\alpha)^2 + (\beta j+\alpha)^2 + (\beta j-\alpha)^2\right) \\
= \ &-3m^2\alpha^2.
\end{aligned}$$

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