K-ORDER LINEAR RECURSIVE SEQUENCES AND THE GOLDEN RATIO

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Abstract. In this paper, we investigate sequences $\{G_{n+1}/G_n\}_{n=1}^{\infty}$ which are approaching the Golden Ratio, where $\{G_n\}_{n=0}^{\infty}$ is a k-order linear recursive sequence of real numbers. We show those cases, where the sequence $\{G_{n+1}/G_n\}_{n=1}^{\infty}$ converges quicker to the Golden Ratio than $\{F_{n+1}/F_n\}_{n=1}^{\infty}$ (F_n denotes the n-th Fibonacci number).

1. Introduction

Let $A_0, A_1, \dots, A_{k-1}$ be given real numbers with $A_{k-1} \neq 0$, where $k \geq 2$ is a fixed integer. A linear recursive sequence $\{G_n\}_{n=0}^{\infty}$ of order k is defined by the recursion

$$G_n = A_0 G_{n-1} + A_1 G_{n-2} + \cdots + A_{k-1} G_{n-k} \qquad (n \geq k), \tag{1.1}$$

where the initial terms $G_0, G_1, \dots, G_{k-1}$ are fixed real numbers with $|G_0|+|G_1|+\cdots+|G_{k-1}| \neq 0$. The polynomial

$$p(x) = x^k - A_0 x^{k-1} - A_1 x^{k-2} - \cdots - A_{k-2} x - A_{k-1} \tag{1.2}$$

is said to be the characteristic polynomial of the sequence $\{G_n\}_{n=0}^{\infty}$. The roots of the equation $p(x) = 0$ are denoted by α_i's $(1 \leq i \leq k)$. In the sequel, we suppose that the root α_1 is of the largest absolute value, that is, $|\alpha_1| > |\alpha_2| \geq \cdots \geq |\alpha_k| > 0$ and the multiplicity of α_1 is 1. According to the literature (see, e.g.,[4], p. 45 and [3], p. 27), α_1 is called the dominant root, and if we denote by m_i the multiplicity of the distinct α_i's $(1 \leq i \leq l, \sum_{i=1}^{l} m_i = k)$ then the Binet formula for the term G_n is as follows

$$G_n = a\alpha_1^n + p_2(n)\alpha_2^n + p_3(n)\alpha_3^n + \cdots + p_l(n)\alpha_l^n, \tag{1.3}$$

where the degree of the polynomial p_i $(2 \leq i \leq l)$ is less than m_i. The constant a and the polynomials p_i belong to the ring $\mathbb{Q}(\alpha_1, \alpha_2, \dots, \alpha_l)[x]$, and we suppose that the initial terms are chosen such that $a \neq 0$ in (1.3). It is known that

$$\lim_{n\to\infty} \frac{G_{n+1}}{G_n} = \alpha_1.$$

Let $n \geq 2$, $F_0 = 0$, $F_1 = 1$, and

$$F_n = F_{n-1} + F_{n-2},$$

which is the well known Fibonacci sequence. The Binet formula for $n \geq 0$ is

$$F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}},$$

where φ and ψ are the roots of the characteristic polynomial of $\{F_n\}_{n=0}^{\infty}$, that is, $p(x) = x^2 - x - 1$, $p(\varphi) = p(\psi) = 0$, $\varphi = \frac{1+\sqrt{5}}{2}$, and $\psi = \frac{1-\sqrt{5}}{2}$, φ is also known as the Golden Ratio and

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi. \tag{1.4}$$

It can be read in [1], p. 346, that if $H_0 = a - b$, $H_1 = b$, and for $n \geq 2$

$$H_n = H_{n-1} + H_{n-2} \tag{1.5}$$

then

$$H_n = aF_{n-1} + bF_{n-2},$$

where $a, b \in \mathbb{R}, b \neq 0$, and

$$\lim_{n\to\infty} \frac{H_{n+1}}{H_n} = \varphi. \tag{1.6}$$

It can be seen that the sequence $\{H_n\}_{n=0}^{\infty}$ differs from the sequence $\{F_n\}_{n=0}^{\infty}$ only in the initial terms and the characteristic polynomial is the same. Note that in [2] similar results are obtained for irrational limits other than the Golden Ratio.

The question then arises, does the limit give a quicker convergence to φ, where the definition of the quicker convergence can be found, e.g., [4], p. 46 and [3], p. 29: Let $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ be convergent sequences of real numbers with $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = z$, we say that $\{y_n\}_{n=0}^{\infty}$ converges quicker than $\{x_n\}_{n=0}^{\infty}$ if

$$\lim_{n\to\infty} \frac{y_n - z}{x_n - z} = 0. \tag{1.7}$$

In this paper, we investigate linear recursive sequences $\{G_n\}_{n=0}^{\infty}$ of real numbers, where the sequences $\left\{\frac{G_{n+1}}{G_n}\right\}_{n=1}^{\infty}$ converge quicker to the Golden Ratio than $\left\{\frac{F_{n+1}}{F_n}\right\}_{n=1}^{\infty}$. (Naturally, we suppose that division by zero never occurs.)

In the following two parts we deal with binary and ternary sequences, while in the last part we investigate the problem for arbitrary k-order sequences.

2. Binary linear recursive sequences and the Golden Ratio

At first, we deal with the special sequence $\{H_n\}_{n=0}^{\infty}$ from (1.5). The Binet formula for $\{H_n\}_{n=0}^{\infty}$ is

$$H_n = \frac{c\varphi^n - d\psi^n}{\sqrt{5}},$$

where $c = b - \psi(a-b) \neq 0$ and $d = b - \varphi(a-b)$. Using (1.4) and (1.6), one can easily verify that

$$\lim_{n\to\infty} \frac{\frac{H_{n+1}}{H_n} - \varphi}{\frac{F_{n+1}}{F_n} - \varphi} = \frac{d}{c}.$$

This implies that the following theorem is true:

Theorem 2.1. *The sequence* $\left\{\frac{H_{n+1}}{H_n}\right\}_{n=1}^{\infty}$ *does not converge quicker to* φ *than the sequence* $\left\{\frac{F_{n+1}}{F_n}\right\}_{n=1}^{\infty}$ *if* $d \neq 0$. *But if* $d = 0$ *then* $\{H_n\}_{n=0}^{\infty}$ *is a simple geometric sequence, where* $\frac{H_{n+1}}{H_n} = \varphi$.

Let us consider now the general second order linear recursive sequence $\{G_n\}_{n=0}^{\infty}$ of real numbers,

$$G_n = A_0 G_{n-1} + A_1 G_{n-2}, \quad (n \geq 2)$$

with its characteristic polynomial

$$p(x) = x^2 - A_0 x - A_1 = (x - \alpha_1)(x - \alpha_2),$$

where $\alpha_1 = \varphi$ is the dominant root, that is, $|\alpha_2| < \varphi$ and $\alpha_2, G_0, G_1 \in \mathbb{R}$. Obviously $A_0 = \alpha_1 + \alpha_2, A_1 = -\alpha_1\alpha_2$. By the Binet formula this sequence has an explicit form for $n \geq 0$,

$$G_n = a\varphi^n + b\alpha_2^n,$$

where a and b are computable constants depending only on the initial terms and the roots, and we suppose that $ab \neq 0$. In this case one can easily verify that

$$\lim_{n\to\infty} \frac{G_{n+1}}{G_n} = \varphi \quad \text{and} \quad \lim_{n\to\infty} \frac{\frac{G_{n+1}}{G_n} - \varphi}{\frac{F_{n+1}}{F_n} - \varphi} = \lim_{n\to\infty} \frac{b(\alpha_2 - \varphi)}{a(\varphi - \psi)} \cdot \left(\frac{\alpha_2}{\psi}\right)^n.$$

This implies that the following theorem is true:

Theorem 2.2. *The sequence* $\left\{\frac{G_{n+1}}{G_n}\right\}_{n=1}^{\infty}$ *converges quicker to* φ *than* $\left\{\frac{F_{n+1}}{F_n}\right\}_{n=1}^{\infty}$ *if and only if* $|\alpha_2| < |\psi|$.

3. Ternary linear recursive sequences and the Golden Ratio

In [1], p. 346, F. Gatta and A. D'Amico investigated the following ternary sequence

$$H_{n+1} = 2H_n - H_{n-2}, \quad (n \geq 3) \tag{3.1}$$

with the real initial terms H_1, H_2, H_3, $(H_3\varphi^2 - H_1\varphi - H_2 \neq 0)$ and using a very special method, they proved that

$$\lim_{n\to\infty} \frac{H_{n+1}}{H_n} = \varphi,$$

that is, they proved that there exist infinitely many third order linear recursive sequences, where the ratio of the consecutive terms tends to the Golden Ratio. But they did not investigate whether this convergence was quicker or not. Applying (1.3), (3.1) has an explicit form for $n \geq 1$,

$$H_n = a\varphi^n + b\psi^n + c, \tag{3.2}$$

where $a = H_3\varphi^2 - H_1\varphi - H_2 \neq 0, b, c$ are computable real constants depending only on the initial terms and the roots of its characteristic polynomial

$$p(x) = x^3 - 2x^2 + 1 = (x^2 - x - 1)(x - 1) = (x - \varphi)(x - \psi)(x - 1).$$

Investigating the following limit we can obtain:

$$\lim_{n\to\infty} \frac{\frac{H_{n+1}}{H_n} - \varphi}{\frac{F_{n+1}}{F_n} - \varphi} = \lim_{n\to\infty} \left(-b + \frac{c(1-\varphi)}{\sqrt{5}\cdot\psi^n}\right) \cdot \frac{1}{a}, \tag{3.3}$$

which implies for the sequence (3.2) that the following result is true.

Theorem 3.1. *Using the notation (3.2):*

- *If* $c \neq 0$ *then the sequence* $\left\{\frac{H_{n+1}}{H_n}\right\}_{n=1}^{\infty}$ *does not converge quicker to* φ *than the sequence* $\left\{\frac{F_{n+1}}{F_n}\right\}_{n=1}^{\infty}$. *More precisely the sequence* $\left\{\frac{F_{n+1}}{F_n}\right\}_{n=1}^{\infty}$ *converges quicker than the sequence* $\left\{\frac{H_{n+1}}{H_n}\right\}_{n=1}^{\infty}$.
- *If* $c = 0$ *and* $b \neq 0$ *then the sequence* $\left\{\frac{H_{n+1}}{H_n}\right\}_{n=1}^{\infty}$ *does not converge quicker to* φ *than the sequence* $\left\{\frac{F_{n+1}}{F_n}\right\}_{n=1}^{\infty}$.
- *If* $c = 0$ *and* $b = 0$ *then* $H_n = a\varphi^n$, *which is a simple geometric sequence.*

Remark 3.2. *The previous theorem dealt with the sequence (3.1) investigated by F. Gatta and A. D'Amico in* [1]*, but – omitting the details – similar results can be obtained in the following case, too:*

$$H_{n+1} = 2H_{n-1} + H_{n-2},$$

where $n \geq 3$ *and* H_1, H_2, H_3 *are initial terms. In this case the characteristic polynomial is*

$$p(x) = x^3 - 2x - 1 = (x^2 - x - 1)(x + 1) = (x - \varphi)(x - \psi)(x + 1)$$

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Let us consider now the general third order linear recursive sequences $\{G_n\}_{n=0}^{\infty}$ of real numbers,

$$G_n = A_0 G_{n-1} + A_1 G_{n-2} + A_2 G_{n-3}$$

with its characteristic polynomial

$$p(x) = x^3 - A_0 x^2 - A_1 x - A_2 = (x - \varphi)(x - \alpha_2)(x - \alpha_3),$$

where $A_0, A_1, A_2 \in \mathbb{R}, (A_2 \neq 0)$, α_2, α_3 are non zero complex numbers, and $|\alpha_2| < \varphi, |\alpha_3| < \varphi$, that is, φ is the dominant root.

Theorem 3.3. *The sequence* $\left\{\frac{G_{n+1}}{G_n}\right\}_{n=1}^{\infty}$ *converges quicker to* φ *than* $\left\{\frac{F_{n+1}}{F_n}\right\}_{n=1}^{\infty}$ *if and only if* $|\alpha_2| < |\psi|$ *and* $|\alpha_3| < |\psi|$.

Proof. We have to investigate the limit (1.7) which will be examined in three different cases.

(i) α_2, α_3 are distinct real numbers. By the Binet formula

$$G_n = a\varphi^n + b\alpha_2^n + c\alpha_3^n.$$

$$\lim_{n\to\infty} \frac{\frac{G_{n+1}}{G_n} - \varphi}{\frac{F_{n+1}}{F_n} - \varphi} = \lim_{n\to\infty} \frac{\frac{G_{n+1} - \varphi G_n}{G_n}}{\frac{F_{n+1} - \varphi F_n}{F_n}} = \lim_{n\to\infty} \frac{G_{n+1} - \varphi G_n}{F_{n+1} - \varphi F_n} \cdot \frac{F_n}{G_n} =$$
$$\lim_{n\to\infty} \frac{a\varphi^{n+1} + b\alpha_2^{n+1} + c\alpha_3^{n+1} - \varphi(a\varphi^n + b\alpha_2^n + c\alpha_3^n)}{\varphi^{n+1} - \psi^{n+1} - \varphi(\varphi^n - \psi^n)} \cdot$$
$$\frac{\varphi^n - \psi^n}{a\varphi^n + b\alpha_2^n + c\alpha_3^n} =$$
$$\lim_{n\to\infty} \frac{b\alpha_2^n(\alpha_2 - \varphi) + c\alpha_3^n(\alpha_3 - \varphi)}{\psi^n(\varphi - \psi)} \cdot \frac{\varphi^n - \psi^n}{a\varphi^n + b\alpha_2^n + c\alpha_3^n} =$$
$$\lim_{n\to\infty} \frac{b\left(\frac{\alpha_2}{\psi}\right)^n(\alpha_2 - \varphi) + c\left(\frac{\alpha_3}{\psi}\right)^n(\alpha_3 - \varphi)}{a\sqrt{5}}.$$

(ii) α_2, α_3 are real numbers and $\alpha_2 = \alpha_3$. By the Binet formula in (1.3)

$$G_n = a\varphi^n + (bn + c)\alpha_2^n.$$

$$\lim_{n\to\infty}\frac{\frac{G_{n+1}}{G_n}-\varphi}{\frac{F_{n+1}}{F_n}-\varphi}=\lim_{n\to\infty}\frac{\frac{G_{n+1}-\varphi G_n}{G_n}}{\frac{F_{n+1}-\varphi F_n}{F_n}}=\lim_{n\to\infty}\frac{G_{n+1}-\varphi G_n}{F_{n+1}-\varphi F_n}\cdot\frac{F_n}{G_n}=$$
$$\lim_{n\to\infty}\frac{a\varphi^{n+1}+(b(n+1)+c)\alpha_2^{n+1}-\varphi(a\varphi^n+(bn+c)\alpha_2^n)}{\varphi^{n+1}-\psi^{n+1}-\varphi(\varphi^n-\psi^n)}\cdot$$
$$\frac{\varphi^n-\psi^n}{a\varphi^n+(bn+c)\alpha_2^n}=$$
$$\lim_{n\to\infty}\frac{(bn+c)\alpha_2^n(\alpha_2-\varphi)+b\alpha_2^{n+1}}{\psi^n(\varphi-\psi)}\cdot\frac{\varphi^n-\psi^n}{a\varphi^n+(bn+c)\alpha_2^n}=$$
$$\lim_{n\to\infty}\frac{(bn+c)\left(\frac{\alpha_2}{\psi}\right)^n(\alpha_2-\varphi)+b\alpha_2\left(\frac{\alpha_2}{\psi}\right)^n}{a\sqrt{5}}.$$

(iii) $\alpha_2=z$, $\alpha_3=\overline{z}$ are non real complex numbers, and by the Binet formula in (1.3)

$$G_n=a\varphi^n+bz^n+c\overline{z}^n.$$

$$\lim_{n\to\infty}\frac{\frac{G_{n+1}}{G_n}-\varphi}{\frac{F_{n+1}}{F_n}-\varphi}=\lim_{n\to\infty}\frac{\frac{G_{n+1}-\varphi G_n}{G_n}}{\frac{F_{n+1}-\varphi F_n}{F_n}}=\lim_{n\to\infty}\frac{G_{n+1}-\varphi G_n}{F_{n+1}-\varphi F_n}\cdot\frac{F_n}{G_n}=$$
$$\lim_{n\to\infty}\frac{a\varphi^{n+1}+bz^{n+1}+c\overline{z}^{n+1}-\varphi(a\varphi^n+bz^n+c\overline{z}^n)}{\varphi^{n+1}-\psi^{n+1}-\varphi(\varphi^n-\psi^n)}\cdot$$
$$\frac{\varphi^n-\psi^n}{a\varphi^n+bz^n+c\overline{z}^n}=$$
$$\lim_{n\to\infty}\frac{bz^n(z-\varphi)+c\overline{z}^n(\overline{z}-\varphi)}{\psi^n(\varphi-\psi)}\cdot\frac{\varphi^n-\psi^n}{a\varphi^n+bz^n+c\overline{z}^n}=$$
$$\lim_{n\to\infty}\frac{b\left(\frac{z}{\psi}\right)^n(z-\varphi)+c\left(\frac{\overline{z}}{\psi}\right)^n(\overline{z}-\varphi)}{a\sqrt{5}}.$$

These limits imply that in all the above cases the limits are equal to zero if and only if in (i) and in (ii) $|\alpha_2|<|\psi|,|\alpha_3|<|\psi|$, while in (iii) $|z|<|\psi|$. Thus our theorem has been proved. □

4. k-order linear recursive sequences and the Golden Ratio

Let us consider now the k-order linear recursive sequence $\{G_n\}_{n=0}^{\infty}$ of real numbers,

$$G_n=A_0G_{n-1}+A_1G_{n-2}+\cdots+A_{k-1}G_{n-k}$$

with its characteristic polynomial

$$p(x)=x^k-A_0x^{k-1}-A_1x^{k-2}-\cdots-A_{k-2}x-A_{k-1}.$$

By (1.3) the Binet formula for term G_n is the following:

$$G_n=a\alpha_1^n+p_2(n)\alpha_2^n+p_3(n)\alpha_3^n+\cdots+p_l(n)\alpha_l^n,$$

where let $\alpha_1=\varphi$ be the dominant root.

Theorem 4.1. *The sequence* $\left\{\frac{G_{n+1}}{G_n}\right\}_{n=1}^{\infty}$ *converges quicker to the Golden Ratio than* $\left\{\frac{F_{n+1}}{F_n}\right\}_{n=1}^{\infty}$, *if* $|\alpha_i| < |\psi|, i = 2, 3, \ldots, l$.

Proof.

$$\lim_{n\to\infty} \frac{\frac{G_{n+1}}{G_n} - \varphi}{\frac{F_{n+1}}{F_n} - \varphi} = \lim_{n\to\infty} \frac{\frac{G_{n+1}-\varphi G_n}{G_n}}{\frac{F_{n+1}-\varphi F_n}{F_n}} = \lim_{n\to\infty} \frac{G_{n+1} - \varphi G_n}{F_{n+1} - \varphi F_n} \cdot \frac{F_n}{G_n} =$$

$$\lim_{n\to\infty} \frac{a\varphi^{n+1} + p_2(n+1)\alpha_2^{n+1} + \cdots + p_l(n+1)\alpha_l^{n+1} - \varphi(a\varphi^n + p_2(n)\alpha_2^n + \cdots + p_l(n)\alpha_l^n)}{\varphi^{n+1} - \psi^{n+1} - \varphi(\varphi^n - \psi^n)} \cdot$$

$$\frac{\varphi^n - \psi^n}{a\varphi^n + \cdots + p_l(n)\alpha_l^n} =$$

$$\lim_{n\to\infty} \frac{p_2(n+1)\alpha_2^{n+1} + \cdots + p_l(n+1)\alpha_l^{n+1} - \varphi p_2(n)\alpha_2^n - \cdots - \varphi p_l(n)\alpha_l^n}{\psi^n\sqrt{5}} \cdot$$

$$\frac{\varphi^n - \psi^n}{a\varphi^n + \cdots + p_l(n)\alpha_l^n} =$$

$$\lim_{n\to\infty} \frac{\alpha_2^n \overbrace{(p_2(n+1)\alpha_2 - p_2(n)\varphi)}^{r_2(n)} + \cdots + \alpha_l^n \overbrace{(p_l(n+1)\alpha_l - p_l(n)\varphi)}^{r_l(n)}}{\psi^n a\sqrt{5}} =$$

$$\lim_{n\to\infty} \frac{\left(\frac{\alpha_2}{\psi}\right)^n r_2(n) + \left(\frac{\alpha_3}{\psi}\right)^n r_3(n) + \cdots + \left(\frac{\alpha_l}{\psi}\right)^n r_l(n)}{a\sqrt{5}}.$$

Investigating different cases of the limit above, one can see that the limit is equal to zero if $\left(\frac{\alpha_i}{\psi}\right)^n$ tends to zero for all $i = 2, 3, \ldots, l$. □

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