CONGRUENCES FOR BERNOULLI - LUCAS SUMS

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Abstract. We give strong congruences for sums of the form $\sum_{n=0}^{N} B_n V_{n+1}$ where B_n denotes the Bernoulli number and V_n denotes a Lucas sequence of the second kind. These congruences, and several variations, are deduced from the reflection formula for p-adic multiple zeta functions.

1. Introduction

In this paper we are concerned with the *Lucas sequences of the second kind* which are defined by the recurrence

$$V_n = PV_{n-1} + QV_{n-2}, \quad V_0 = 2, \quad V_1 = P, \tag{1.1}$$

where P and Q are integers. (If the initial conditions were $V_0 = 0$, $V_1 = 1$ the sequence is called a *Lucas sequence of the first kind*, see (5.1) below.) Our main result (Corollary 3.2 below) is that the series

$$\sum_{n=0}^{\infty} B_n^{(r)} (P/Qr)^{n+1} V_{n+1} = 0 \quad \text{in} \quad \mathbb{Q}_p \tag{1.2}$$

for all primes p dividing the numerator of (P^2/Qr^2), where $\mathbb{Q}_p$ denotes the field of p-adic numbers and $B_n^{(r)}$ denotes the Bernoulli number of order r, defined below. The fact that these series converge to zero in $\mathbb{Q}_p$ will be used to deduce congruences for their partial sums, such as

$$\sum_{n=0}^{2N} B_n (P/Q)^{n+1} V_{n+1} \equiv 0 \pmod{p^{N+1}} \tag{1.3}$$

for all primes p dividing the numerator of (P^2/Q), meaning that each such partial sum is a rational number whose numerator is divisible by p^{N+1}. The main result is a consequence of the reflection formula for p-adic multiple zeta functions. We conclude with many variations on this theme.

2. Notations and preliminaries

The sequence $\{V_n\}$ defined by (1.1) satisfies the well-known Binet formula

$$V_n = a^n + b^n \tag{2.1}$$

where $a, b = (P \pm \sqrt{P^2 + 4Q})/2$ are the reciprocal roots of the characteristic polynomial $f(T) = 1 - PT - QT^2 = (1 - aT)(1 - bT)$. Clearly we have $a + b = P$ and $ab = -Q$.

The *order r Bernoulli polynomials* $B_n^{(r)}(x)$ are defined [6, 3] by

$$\left(\frac{t}{e^t - 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}; \tag{2.2}$$

these are polynomials of degree n in x and their values at $x = 0$ are the *Bernoulli numbers of order* r, $B_n^{(r)} := B_n^{(r)}(0)$. When $r = 1$ we have the usual Bernoulli numbers $B_n := B_n^{(1)}(0)$. It is well-known that $B_{2n+1} = 0$ for positive integers n; the denominator of B_{2n} is squarefree, being equal to the product of those primes p such that $p-1$ divides $2n$ (von Staudt - Clausen Theorem). Therefore the denominator of every even-indexed Bernoulli number is a multiple of 6.

We now summarize the basics of p-adic (Barnes type) multiple zeta functions [8]. For a prime number p we use $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ to denote the ring of p-adic integers, the field of p-adic numbers, and the completion of an algebraic closure of $\mathbb{Q}_p$, respectively. Let $|\cdot|_p$ denote the unique absolute value defined on $\mathbb{C}_p$ normalized by $|p|_p = p^{-1}$. Given $a \in \mathbb{C}_p^\times$, we define the p-adic valuation $\nu_p(a) \in \mathbb{Q}$ to be the unique exponent such that $|a|_p = p^{-\nu_p(a)}$. By convention we set $\nu_p(0) = \infty$.

We choose an embedding of the algebraic closure $\bar{\mathbb{Q}}$ into $\mathbb{C}_p$ and fix it once and for all. Let $p^{\mathbb{Q}}$ denote the image in $\mathbb{C}_p^\times$ of the set of positive real rational powers of p under this embedding. Let μ denote the group of roots of unity in $\mathbb{C}_p^\times$ of order not divisible by p. If $a \in \mathbb{C}_p$, $|a|_p = 1$ then there is a unique element $\hat{a} \in \mu$ such that $|a - \hat{a}|_p < 1$ (called the *Teichmüller representative* of a); it may also be defined analytically by $\hat{a} = \lim_{n\to\infty} a^{p^{n!}}$. We extend this definition to $a \in \mathbb{C}_p^\times$ by

$$\hat{a} = (\widehat{a/p^{\nu_p(a)}}), \tag{2.3}$$

that is, we define $\hat{a} = \hat{u}$ if $a = p^r u$ with $p^r \in p^{\mathbb{Q}}$ and $|u|_p = 1$. We then define the function $\langle\cdot\rangle$ on $\mathbb{C}_p^\times$ by $\langle a\rangle = p^{-\nu_p(a)} a/\hat{a}$. This yields an internal direct product decomposition of multiplicative groups

$$\mathbb{C}_p^\times \simeq p^{\mathbb{Q}} \times \mu \times D \tag{2.4}$$

where $D = \{a \in \mathbb{C}_p : |a-1|_p < 1\}$, given by

$$a = p^{\nu_p(a)} \cdot \hat{a} \cdot \langle a\rangle \mapsto (p^{\nu_p(a)}, \hat{a}, \langle a\rangle). \tag{2.5}$$

In [8] we defined p-adic multiple zeta functions $\zeta_{p,r}(s,a)$ for $r \in \mathbb{Z}^+$ and $a \in \mathbb{C}_p \setminus \mathbb{Z}_p$ by an r-fold Volkenborn integral. However, for the purposes of this paper, we will only be concerned with the case where $|a|_p > 1$, and we will take the series

$$\zeta_{p,r}(s,a) = \frac{a^r \langle a\rangle^{-s}}{(s-1)\cdots(s-r)} \sum_{n=0}^{\infty} \binom{r-s}{n} B_n^{(r)} a^{-n} \tag{2.6}$$

([8], Theorem 4.1) as the definition of $\zeta_{p,r}(s,a)$ for positive integers r; this series is convergent for $s \in \mathbb{Z}_p$ when $|a|_p > 1$, and defines $\zeta_{p,r}(s,a)$ as a C^∞ function of $s \in \mathbb{Z}_p \setminus \{1, 2, ..., r\}$ and a locally analytic function of a for $|a|_p > 1$. (This is more than sufficient for our purposes; for a complete discussion of continuity and analyticity of $\zeta_{p,r}(s,a)$ see [8], [11]). It will be seen that for $|a|_p > 1$, the values at the negative integers are given by

$$\zeta_{p,r}(-m,a) = \frac{(-1)^r r!}{(m+r)!} \left(\frac{\langle a\rangle}{a}\right)^m B_{m+r}^{(r)}(a) \tag{2.7}$$

([8], Theorem 3.2(v)). The p-adic multiple zeta functions satisfy many identities; the important one for our present purposes is the reflection formula, which reads

$$\zeta_{p,r}(s,a) = (-1)^r \langle -1\rangle^{-s} \zeta_{p,r}(s, r-a) \tag{2.8}$$

([8], Theorem 3.2; [11],eq.(2.18)). Note that for odd primes p we have $\langle -1\rangle = 1$; for $p = 2$ we have $\langle -1\rangle = -1$. The reflection formula for $\zeta_{p,r}(s,a)$ arises from the reflection formula

$$B_n^{(r)}(r-a) = (-1)^n B_n^{(r)}(a) \tag{2.9}$$

for the Bernoulli polynomials; specifically, from (2.9) and (2.7) we observe that (2.8) holds when s is a negative integer; but both sides are continuous and the negative integers are dense in $\mathbb{Z}_p$, so it holds for all $s \in \mathbb{Z}_p$.

3. Bernoulli - Lucas series

We begin this section with the simplest case of our class (1.2) of series, and then expand from there.

Theorem 3.1. *Let $r, k \in \mathbb{Z}$ with $r > 0$ and let $\{V_n\}$ denote the Lucas sequence of the second kind defined by the recurrence*

$$V_n = rkV_{n-1} + kV_{n-2}, \quad V_0 = 2, \quad V_1 = rk.$$

Then the series

$$\sum_{n=0}^{\infty} B_n^{(r)} V_{n+1} = 0 \quad in \quad \mathbb{Q}_p$$

for all primes p dividing k.

Proof. From the Laurent series expansion (2.6) with $s = r+1$ we observe

$$\zeta_{p,r}(r+1,x) = -\frac{1}{r!}\left(\frac{x}{\langle x\rangle}\right)^{r+1} \sum_{n=0}^{\infty} B_n^{(r)}(-x)^{-n-1} \tag{3.1}$$

for $|x|_p > 1$, since $\binom{-1}{n} = (-1)^n$. If we set $x = -1/a$, then $r - x = (ra+1)/a$ and the reflection formula (2.8) implies

$$\begin{aligned} 0 &= \zeta_{p,r}(r+1,x) + (-1)^{r+1}\langle -1\rangle^{-(r+1)}\zeta_{p,r}(r+1,r-x) \\ &= -\frac{1}{r!}\left(\left(\frac{x}{\langle x\rangle}\right)^{r+1}\sum_{n=0}^{\infty} B_n^{(r)} a^{n+1} + \left(\frac{x-r}{\langle x-r\rangle}\right)^{r+1}\sum_{n=0}^{\infty} B_n^{(r)}\left(\frac{-a}{ra+1}\right)^{n+1}\right) \\ &= -\frac{1}{r!}\left(\frac{x}{\langle x\rangle}\right)^{r+1}\sum_{n=0}^{\infty} B_n^{(r)}\left(a^{n+1}+b^{n+1}\right) \end{aligned} \tag{3.2}$$

where $b = -a/(ra+1)$. In the above calculation we have used the fact that $\langle\cdot\rangle$ is a multiplicative homomorphism and that $\frac{\langle x+y\rangle}{x+y} = \frac{\langle x\rangle}{x}$ whenever $|y|_p < |x|_p$.

Since $a + b = -rab$, we may choose a, b to be the reciprocal roots of the characteristic polynomial $f(T) = 1 - rkT - kT^2 = (1-aT)(1-bT)$, which satisfy $a + b = rk$ and $ab = -k$. By the Binet formula, $V_n = a^n + b^n$ for all n. The condition $|x|_p > 1$ is equivalent to $|a|_p < 1$, which is equivalent to $|k|_p < 1$, which means that p divides k. This completes the proof. □

It will be observed that the condition that $k \in \mathbb{Z}$ in the above theorem is unnecessary; the theorem would remain valid for rational numbers k whose numerator is divisible by p, or indeed for any p-adic number $k \in \mathbb{C}_p$ with $|k|_p < 1$. By means of a simple transformation the above theorem may be made to accommodate almost any Lucas sequence of the second kind.

Corollary 3.2. *Let $r, P, Q \in \mathbb{Z}$ with $r > 0$ and let $\{V_n\}$ denote the Lucas sequence of the second kind defined by the recurrence*

$$V_n = PV_{n-1} + QV_{n-2}, \quad V_0 = 2, \quad V_1 = P.$$

Then the series

$$\sum_{n=0}^{\infty} B_n^{(r)} (P/Qr)^{n+1} V_{n+1} = 0 \quad in \quad \mathbb{Q}_p$$

for all primes p dividing the numerator of (P^2/Qr^2).

Proof. The substitution $v_n = (P/Qr)^n V_n$ transforms the Lucas sequence recurrence

$$V_n = PV_{n-1} + QV_{n-2}, \quad V_0 = 2, \quad V_1 = P \tag{3.3}$$

into the recurrence

$$v_n = (P^2/Qr)v_{n-1} + (P^2/Qr^2)v_{n-2}, \quad v_0 = 2, \quad v_1 = (P^2/Qr). \tag{3.4}$$

The corollary follows by applying the above theorem to $\{v_n\}$ with $k = (P^2/Qr^2)$. □

It will be observed that conditions of the above corollary require the rational number (P^2/Qr^2) to have a numerator other than $\{0, 1, -1\}$, but this is the only requirement for the result to be nontrivial. The corollary may be restated as follows: Whenever the series

$$\sum_{n=0}^{\infty} B_n^{(r)} (P/Qr)^{n+1} V_{n+1} \tag{3.5}$$

converges in $\mathbb{Q}_p$, it converges to zero.

4. Congruences for Bernoulli - Lucas sums

In this section we show how Corollary 3.2 implies congruences for the partial sums of these Bernoulli - Lucas series; for simplicity we consider the case where $r = 1$. As in the proof of Corollary 3.2, the substitution $v_n = (P/Q)^n V_n$ transforms the Lucas sequence recurrence

$$V_n = PV_{n-1} + QV_{n-2}, \quad V_0 = 2, \quad V_1 = P \tag{4.1}$$

into the recurrence

$$v_n = (P^2/Q)v_{n-1} + (P^2/Q)v_{n-2}, \quad v_0 = 2, \quad v_1 = (P^2/Q). \tag{4.2}$$

We put $k = (P^2/Q)$ and suppose the prime p divides the numerator of k.

Proposition 4.1. *Consider the Lucas sequence of the second kind*

$$V_n = kV_{n-1} + kV_{n-2}, \quad V_0 = 2, \quad V_1 = k,$$

where $k \in \mathbb{Q}$ and $\nu_p(k) = e > 0$. Then
i. If p is odd, then $\nu_p(V_{2m}) = me$ and $\nu_p(V_{2m-1}) \geq me$;
ii. If $e > 1$, then $\nu_2(V_{2m}) = me + 1$ and $\nu_2(V_{2m-1}) = me$;
iii. If $e = 1$, then $\nu_2(V_{4m}) = 2m + 1$, $\nu_2(V_{4m+2}) > 2m + 2$, and $\nu_2(V_{2m-1}) = m$.

Proof. These follow by induction on m, using the non-archimedean property of ν_p that $\nu_p(x + y) \geq \min\{\nu_p(x), \nu_p(y)\}$, with equality when $\nu_p(x) \neq \nu_p(y)$. When $p = 2$, we must observe that for $x, y \in \mathbb{Q}_2$ we have $\nu_2(x + y) \geq \min\{\nu_2(x), \nu_2(y)\}$ with equality *if and only if* $\nu_2(x) \neq \nu_2(y)$. □

Theorem 4.2. *Consider the Lucas sequence of the second kind*

$$V_n = kV_{n-1} + kV_{n-2}, \quad V_0 = 2, \quad V_1 = k,$$

where $k \in \mathbb{Q}$ *and* $\nu_p(k) = e > 0$ *for some prime* p. *Then for all positive integers* N,

$$\sum_{n=0}^{2N} B_n V_{n+1} \equiv 0 \qquad (\text{mod}\, p^{(N+2)e-1}).$$

If $p = 2$ *the power of 2 in this congruence is exact, that is,*

$$\nu_2\left(\sum_{n=0}^{2N} B_n V_{n+1}\right) = (N+2)e - 1.$$

Proof. Since the series converges to zero in $\mathbb{Q}_p$, we have

$$\sum_{n=0}^{2N} B_n V_{n+1} = -\sum_{n=2N+2}^{\infty} B_n V_{n+1} \quad \text{in} \quad \mathbb{Q}_p \tag{4.3}$$

by virtue of the fact that $B_n = 0$ for odd $n > 1$. Therefore

$$\nu_p\left(\sum_{n=0}^{2N} B_n V_{n+1}\right) \geq \min_{n \geq 2N+2} \{\nu_p(B_n V_{n+1})\}. \tag{4.4}$$

From the above proposition, all such $\nu_p(V_{n+1})$ on the right side of (4.4) are at least $(N+2)e$, and from the von Staudt-Clausen theorem we have $\nu_p(B_{2n}) \geq -1$ for all n, since the denominator of B_{2n} is squarefree. The first statement follows immediately. For the second statement, the von Staudt-Clausen theorem implies $\nu_p(B_{2n}) = -1$ for all n when $p = 2$ or $p = 3$. Therefore $\nu_2(B_{2n}V_{2n+1}) = (n+1)e - 1$ for all n, so all the ν_p values on the right side of (4.4) are distinct, so the ν_p value of the sum on the left side of (4.4) is exactly equal to their minimum. $\square$

Corollary 4.3. *Consider the Lucas sequence of the second kind*

$$V_n = PV_{n-1} + QV_{n-2}, \quad V_0 = 2, \quad V_1 = P,$$

where $P, Q \in \mathbb{Z}$ *and* $\nu_p(P^2/Q) = e > 0$ *for some prime* p. *Then for all positive integers* N,

$$\sum_{n=0}^{2N} B_n (P/Q)^{n+1} V_{n+1} \equiv 0 \qquad (\text{mod}\, p^{(N+2)e-1}).$$

If $p = 2$ *the power of 2 in this congruence is exact, that is,*

$$\nu_2\left(\sum_{n=0}^{2N} B_n (P/Q)^{n+1} V_{n+1}\right) = (N+2)e - 1.$$

Since the series in question are p-adically convergent, it is clear that the p-adic ordinals of the terms are tending to infinity; but the fact that the series are converging to *zero* shows that the partial sums are also exhibiting an unusual *synergy* in that the p-adic ordinal of each partial sum is typically larger than that of any of its nonzero summands.

Example. Consider the *Lucas numbers* L_n defined by (1.1) with $(P, Q) = (1, 1)$; the sequence begins with the values

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, 2207, \ldots \tag{4.5}$$

Since $P^2/Q = 1$, there is no prime p which satisfies the hypotheses of Corollary 4.3 for L_n. However, the sequence $V_n = L_{2n}$ satisfies (1.1) with $(P, Q) = (3, -1)$, and therefore from Corollary 4.3 we have

$$\sum_{n=0}^{2N} B_n(-3)^{n+1} L_{2n+2} \equiv 0 \pmod{3^{2N+3}}. \tag{4.6}$$

for all positive integers N. In general, for any positive integer m the sequence $V_n = L_{mn}$ satisfies (1.1) with $(P, Q) = (L_m, (-1)^{m+1})$, and therefore by applying Corollary 4.3 to each prime factor p of L_m we obtain

$$\sum_{n=0}^{2N} B_n((-1)^{m+1} L_m)^{n+1} L_{m(n+1)} \equiv 0 \pmod{L_m^{2N+3}}. \tag{4.7}$$

for all positive integers N.

Example. Consider the *Pell-Lucas numbers* V_n defined by (1.1) with $(P, Q) = (2, 1)$; the sequence begins with the values

$$2, 2, 6, 14, 34, 82, 198, 478, 1154, 2786, 6726, 16238, 39202, 94642, 228486, \ldots \tag{4.8}$$

From Corollary 4.3 we have

$$\nu_2\left(\sum_{n=0}^{2N} B_n 2^{n+1} V_{n+1}\right) = 2N + 3 \tag{4.9}$$

for all positive integers N.

Example. Consider the *Lucas-balancing numbers* C_n defined by

$$C_n = 6C_{n-1} - C_{n-2}, \quad C_0 = 1, \quad C_1 = 3, \tag{4.10}$$

which begins with the values

$$1, 3, 17, 99, 577, 3363, 19601, 114243, 665857, 3880899, 22619537, \ldots \tag{4.11}$$

We see that $V_n = 2C_n$ satisfies the recurrence (1.1) with $(P, Q) = (6, -1)$. Applying Corollary 4.3 with both $p = 2$ and $p = 3$ we obtain

$$\sum_{n=0}^{2N} B_n(-6)^{n+1} C_{n+1} \equiv 0 \pmod{3 \cdot 6^{2N+2}}. \tag{4.12}$$

for all positive integers N.

Example. Taking $P = Q = -4$ in (1.1) yields $V_n = 2(-2)^n$. From Corollary 4.3 we have

$$\nu_2\left(\sum_{n=0}^{2N} B_n(-2)^n\right) = 2N + 1 \tag{4.13}$$

for all positive integers N.

Remark. Zagier [12] considered the ordinary generating function $\beta(x) = \sum_{n=0}^{\infty} B_n x^n$ formally, even though it doesn't converge for any $x \neq 0$ (in a real or complex sense). For any prime p, $\beta(x)$ converges in $\mathbb{C}_p$ for $|x|_p < 1$; in this way the functional equation ([12], Prop. A.2) is precisely the difference equation ([8], Theorem 3.2(i)) for $\zeta_{p,1}(s, a)$. The above example says that $\beta(-2) = 0$ in $\mathbb{Q}_2$. This is the only root of $\beta(x)$ in $\mathbb{Q}_p$ for any prime p.

Example. Taking $P = Q = -2$ in (1.1) yields the sequence $\{V_n\}$ which begins with the values

$$2, -2, 0, 4, -8, 8, 0, -16, 32, -32, 0, 64, -128, 128, 0, -256, 512, -512, 0, \ldots \tag{4.14}$$

It can be verified by induction that the odd-index values satisfy $V_{2m-1} = -2^m(-1)^{m(m-1)/2}$, so from Corollary 4.3 we have

$$\nu_2\left(\sum_{m=0}^{N} B_{2m}2^m(-1)^{m(m+1)/2}\right) = N \tag{4.15}$$

for all positive integers N.

Example. Taking $P = Q = -3$ in (1.1) yields the sequence $\{V_n\}$ which begins with the values

$$2, -3, 3, 0, -9, 27, -54, 81, -81, 0, 243, -729, 1458, -2187, 2187, 0, ... \tag{4.16}$$

It can be verified by induction that the odd-index values satisfy $V_{6m-3} = 0$, $V_{6m-1} = -(-27)^m$, and $V_{6m+1} = -3(-27)^m$ for positive integers m. Since $B_0V_1 + B_1V_2 = -9/2$, from Corollary 4.3 we have

$$\nu_3\left(\frac{9}{2} + \sum_{m=1}^{N}(-27)^m\Big(B_{6m-2} + 3B_{6m}\Big)\right) = 3N + 2 \tag{4.17}$$

for all positive integers N.

We remark that one can use Corollary 3.2 and Proposition 4.1 to give similar systems of congruences involving higher order Bernoulli numbers $B_n^{(r)}$ for $r > 1$. The main difference is that $\nu_p(B_n^{(r)})$ is not known as explicitly when $r > 1$; in particular, the property $B_{2n+1}^{(1)} = 0$ does not extend to order $r > 1$.

5. Bernoulli - Lucas sums of the first kind

If we evaluate the p-adic multiple zeta functions $\zeta_{p,r}(s, a)$ at $s = r + 2$ instead of $s = r + 1$, we obtain similar identities involving the *Lucas sequences of the first kind* which are defined by the recurrence

$$U_n = PU_{n-1} + QU_{n-2}, \quad U_0 = 0, \quad U_1 = 1. \tag{5.1}$$

The LSFK satisfy the well-known Binet formula

$$U_n = \begin{cases} \dfrac{a^n - b^n}{a - b}, & \text{if } P^2 + 4Q \neq 0, \\ na^{n-1}, & \text{if } P^2 + 4Q = 0. \end{cases} \tag{5.2}$$

Theorem 5.1. *Let $r, k \in \mathbb{Z}$ with $r > 0$ and let $\{U_n\}$ denote the Lucas sequence of the first kind defined by the recurrence*

$$U_n = rkU_{n-1} + kU_{n-2}, \quad U_0 = 0, \quad U_1 = 1.$$

Then the series

$$\sum_{n=0}^{\infty}(n+1)B_n^{(r)}U_{n+2} = 0 \quad \textit{in} \quad \mathbb{Q}_p$$

for all primes p dividing k.

Proof. From the Laurent series expansion (2.6) with $s = r + 2$ we observe

$$\zeta_{p,r}(r+2, x) = \frac{1}{(r+1)!}\left(\frac{x}{\langle x\rangle}\right)^{r+2}\sum_{n=0}^{\infty}(n+1)B_n^{(r)}(-x)^{-n-2} \tag{5.3}$$

for $|x|_p > 1$, since $\binom{-2}{n} = (-1)^n(n+1)$. If we set $x = -1/a$, then $r - x = (ra+1)/a$ and the reflection formula (2.8) implies

$$\begin{aligned} 0 &= \zeta_{p,r}(r+2,x) - (-1)^{r+2}\langle -1\rangle^{-(r+2)}\zeta_{p,r}(r+2,r-x) \\ &= \frac{1}{(r+1)!}\left(\frac{x}{\langle x\rangle}\right)^{r+2}\sum_{n=0}^{\infty}(n+1)B_n^{(r)}\left(a^{n+2} - b^{n+2}\right) \end{aligned} \tag{5.4}$$

where $b = -a/(ra+1)$. Since $a + b = -rab$, we may choose a, b to be the reciprocal roots of the characteristic polynomial $f(T) = 1 - rkT - kT^2 = (1-aT)(1-bT)$, which satisfy $a + b = rk$ and $ab = -k$. By the Binet formula, $U_n = (a^n - b^n)/(a-b)$ if $r^2k^2 + 4k \neq 0$; in this case the theorem follows by dividing by $a - b$. In the case where $r^2k^2 + 4k = 0$, we start from the general case with $r^2k^2 + 4k \neq 0$, divide by $a - b$, and take the limit as k approaches $-4/r^2$ p-adically. □

Corollary 5.2. *Let $r, P, Q \in \mathbb{Z}$ with $r > 0$ and let $\{U_n\}$ denote the Lucas sequence of the first kind defined by the recurrence*

$$U_n = PU_{n-1} + QU_{n-2}, \quad U_0 = 0, \quad U_1 = 1.$$

Then the series

$$\sum_{n=0}^{\infty}(n+1)B_n^{(r)}(P/Qr)^{n+1}U_{n+2} = 0 \quad in \quad \mathbb{Q}_p$$

for all primes p dividing the numerator of (P^2/Qr^2).

Proof. The substitution $u_n = (P/Qr)^{n-1}U_n$ transforms the Lucas sequence recurrence

$$U_n = PU_{n-1} + QU_{n-2}, \quad U_0 = 0, \quad U_1 = 1 \tag{5.5}$$

into the recurrence

$$u_n = (P^2/Qr)u_{n-1} + (P^2/Qr^2)u_{n-2}, \quad u_0 = 0, \quad u_1 = 1. \tag{5.6}$$

The corollary follows by applying the above theorem to $\{u_n\}$ with $k = (P^2/Qr^2)$. □

Corollary 5.3. *Consider the Lucas sequence of the first kind*

$$U_n = PU_{n-1} + QU_{n-2}, \quad U_0 = 0, \quad U_1 = 1,$$

where $P, Q \in \mathbb{Z}$ and $\nu_p(P^2/Q) = e > 0$ for some prime p. Then for all positive integers N,

$$\sum_{n=0}^{2N}(n+1)B_n(P/Q)^{n+1}U_{n+2} \equiv 0 \qquad (\mathrm{mod}\, p^{(N+2)e-1}).$$

Proof. First treat the case where $P = k$ and $Q = k$, using the facts that $\nu_p(U_{2m}) \geq me$ and $\nu_p(U_{2m+1}) = me$, as in the proof of Theorem 4.2. Then use the substitution $u_n = (P/Q)^{n-1}U_n$ to treat the general case as in Corollary 4.3. □

Example. Consider the *Fibonacci numbers* F_n defined by (5.1) with $(P,Q) = (1,1)$; the sequence begins with the values

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, \ldots \tag{5.7}$$

Since $P^2/Q = 1$, there is no prime p which satisfies the hypotheses of Corollary 5.3 for F_n. However, the sequence $U_n = F_{2n}$ satisfies (5.1) with $(P,Q) = (3,-1)$, and therefore from Corollary 5.3 we have

$$\sum_{n=0}^{2N}(n+1)B_n(-3)^{n+1}F_{2n+4} \equiv 0 \pmod{3^{2N+3}}. \tag{5.8}$$

for all positive integers N. In general, for any positive integer m the sequence F_{mn} is F_m times the LSFK which satisfies (5.1) with $(P,Q) = (L_m,(-1)^{m+1})$, and therefore by applying Corollary 5.3 to each prime factor p of L_m we obtain

$$\sum_{n=0}^{2N}(n+1)B_n((-1)^{m+1}L_m)^{n+1}F_{m(n+2)} \equiv 0 \pmod{F_m L_m^{2N+3}} \tag{5.9}$$

for all positive integers N.

Example. Consider the *Pell numbers* P_n defined by (5.1) with $(P,Q) = (2,1)$; the sequence begins with the values

$$0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, 80782, ... \tag{5.10}$$

From Corollary 5.3 we have

$$\sum_{n=0}^{2N}(n+1)B_n 2^{n+1}P_{n+2} \equiv 0 \pmod{2^{2N+3}} \tag{5.11}$$

for all positive integers N.

Example. Consider the *balancing numbers* U_n defined by (5.1) with $(P,Q) = (6,-1)$; the sequence begins with the values

$$0, 1, 6, 35, 204, 1189, 6930, 40391, 235416, 1372105, 7997214, ... \tag{5.12}$$

From Corollary 5.3 we have

$$\sum_{n=0}^{2N}(n+1)B_n(-6)^{n+1}U_{n+2} \equiv 0 \pmod{6^{2N+3}} \tag{5.13}$$

for all positive integers N.

Examples. Taking the Lucas sequences of the first kind with $(P,Q) = (-4,-4)$, $(-2,-2)$, and $(-3,-3)$, respectively, produces

$$\sum_{n=0}^{2N}(n+1)(n+2)B_n(-2)^n \equiv 0 \pmod{2^{2N+2}}; \tag{5.14}$$

$$\nu_2\left(1 + \sum_{m=0}^{N}(4m+1)B_{4m}(-4)^m\right) = 2N+1; \qquad \text{and} \tag{5.15}$$

$$\nu_3\left(2 + \sum_{m=0}^{N}(-27)^m\Big((6m+1)B_{6m} + 3(6m+3)B_{6m+2}\Big)\right) = 3N+2 \tag{5.16}$$

for all positive integers N. In the last two cases the 2-adic (resp. 3-adic) ordinal of the sum can be determined exactly because the ordinals of the terms can easily be determined exactly.

One can continue this theme by evaluating $\zeta_{p,r}(s,a)$ at $s = r + k$ for any positive integer k; the general result is that

$$\sum_{n=0}^{\infty} \binom{n+k-1}{k-1} B_n^{(r)} (P/Qr)^{n+1} a_{n+k} = 0 \quad \text{in} \quad \mathbb{Q}_p \tag{5.17}$$

for all primes p dividing the numerator of (P^2/Qr^2), where $a_n = V_n$ if k is odd and $a_n = U_n$ if k is even.

6. Euler - Lucas and Stirling - Lucas series

In this final section we mention some further variations of these results which can be obtained involving other sequences related to the Bernoulli numbers. The *order r Euler polynomials* $E_n^{(r)}(x)$ are defined by

$$\left(\frac{2}{e^t+1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!}; \tag{6.1}$$

these are polynomials of degree n in x and their values at $x = 0$ are the *order r Euler numbers* $E_n^{(r)} := E_n^{(r)}(0)$. In a manner analogous to Theorems 3.1 and 5.1 and their corollaries, one may also prove

$$\sum_{n=0}^{\infty} E_n^{(r)} (P/Qr)^{n+1} V_{n+1} = 0 \quad \text{in} \quad \mathbb{Q}_p \tag{6.2}$$

and

$$\sum_{n=0}^{\infty} (n+1) E_n^{(r)} (P/Qr)^{n+1} U_{n+2} = 0 \quad \text{in} \quad \mathbb{Q}_p \tag{6.3}$$

for all primes p dividing the numerator of (P^2/Qr^2), where U_n and V_n denote the Lucas sequences (5.1) and (1.1). This can be proved by considering the p-adic function

$$\eta_{p,r}(s,a) = \langle a \rangle^{-s} \sum_{n=0}^{\infty} \binom{-s}{n} E_n^{(r)} a^{-n} \tag{6.4}$$

for $|a|_p > 1$ and $s \in \mathbb{Z}_p$. (We observe from ([10], Theorem 3.2) that $\nu_p(E_n^{(r)}) \geq 0$ for all n and r when p is odd. For $p = 2$, we note that

$$E_n^{(1)} = 2(1 - 2^{n+1}) \frac{B_{n+1}}{n+1} \tag{6.5}$$

so that

$$\nu_2(E_n^{(1)}) = \begin{cases} 0, & \text{if } n = 0, \\ \infty, & \text{if } n > 0 \text{ is even}, \\ -\nu_2(n+1), & \text{if } n \text{ is odd}; \end{cases} \tag{6.6}$$

then from

$$E_n^{(r)} = \sum_{n_1+\cdots+n_r=n} \binom{n}{n_1, \ldots, n_r} E_{n_1}^{(1)} \cdots E_{n_r}^{(1)} \tag{6.7}$$

we may obtain the crude bound $\nu_2(E_n^{(r)}) \geq -r \log_2(n+1)$. This is enough to show that for $|a|_p > 1$, the series in (6.4) is a uniformly convergent series, for $s \in \mathbb{Z}_p$, of polynomials $\binom{-s}{n}$

which are $\mathbb{Z}_p$-valued for $s \in \mathbb{Z}_p$, and thus represents a C^∞ function of $s \in \mathbb{Z}_p$.) At the negative integers, we see that

$$\eta_{p,r}(-m,a) = \left(\frac{\langle a\rangle}{a}\right)^m E_m^{(r)}(a) \tag{6.8}$$

which implies that we can express $\eta_{p,1}(s,a) = 2\Phi_p(-1,s,a)$ in terms of the p-adic Lerch transcendent Φ_p defined in [11], or $\eta_{p,1}(s,a) = \zeta_{p,E}(s-1,a)$ in terms of the p-adic Euler zeta function defined in [5]. From the reflection formula

$$E_n^{(r)}(r-a) = (-1)^n E_n^{(r)}(a) \tag{6.9}$$

for Euler polynomials we obtain the reflection formula

$$\eta_{p,r}(s,r-a) = \langle -1\rangle^{-s}\eta_{p,r}(s,a) \tag{6.10}$$

for the p-adic function $\eta_{p,r}$. This is a generalization of the reflection formula ([5], Theorem 3.10) for the function $\zeta_{p,E}(s,a)$. The results (6.2), (6.3) then follow by evaluating $\eta_{p,r}(s,a)$ at $s=1$ and $s=2$, respectively, using this reflection formula (6.10). In general, one can evaluate $\eta_{p,r}(s,a)$ at $s=k$ for any positive integer k and obtain a result analogous to (5.17).

Finally, one may use the *negative integer order* p-adic zeta functions $\zeta_{p,-r}(s,a)$ to produce similar series involving the *Stirling numbers of the second kind* $S(n,r) := S(n,r|0)$, where

$$(e^t-1)^r e^{xt} = r!\sum_{n=r}^{\infty} S(n,r|x)\frac{t^n}{n!} \tag{6.11}$$

generates the weighted Stirling numbers of the second kind [1, 2] with weight x. The analogous series obtained are

$$\sum_{n=r}^{\infty} S(n,r)(-P/Qr)^{n+1}V_{n+1} = 0 \quad \text{in} \quad \mathbb{Q}_p \tag{6.12}$$

for even r, where V_n is given by (1.1) and p divides the numerator of (P^2/Qr^2); and

$$\sum_{n=r}^{\infty} S(n,r)(-P/Qr)^{n}U_{n+1} = 0 \quad \text{in} \quad \mathbb{Q}_p \tag{6.13}$$

for odd r, where U_n is given by (5.1) and p divides the numerator of (P^2/Qr^2). We take a positive integer r and consider the p-adic function defined by

$$\zeta_{p,-r}(s,a) = a^{-r}\langle a\rangle^{-s}s(s+1)\cdots(s+r-1)\sum_{n=0}^{\infty}\binom{-r-s}{n}B_n^{(-r)}a^{-n} \tag{6.14}$$

for $|a|_p > 1$ and $s \in \mathbb{Z}_p$. Using the identity

$$B_n^{(-r)} = \binom{n+r}{r}^{-1} S(n+r,r) \tag{6.15}$$

we find that

$$\zeta_{p,-r}(-m,a) = (-1)^r\left(\frac{\langle a\rangle}{a}\right)^m r!S(m,r|a) \tag{6.16}$$

for all nonnegative integers m; this shows that these functions agree with the functions $\zeta_{p,-r}(s,a)$ defined in [11]. We appeal to the reflection formula

$$\zeta_{p,-r}(s,-r-a) = (-1)^r\langle -1\rangle^{-s}\zeta_{p,-r}(s,a) \tag{6.17}$$

([11], eq. (3.5)) and evaluate the function $\zeta_{p,-r}(s,a)$ at $s=1$ to obtain the results.

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