ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
FLORIAN LUCA

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PROBLEMS PROPOSED IN THIS ISSUE

H-760 Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania.

Prove that if $m \geq 1$, $k \geq 1$, $n \geq 0$ are integers then

$$m^m \sum_{p=0}^{2n+1} \left(1 + \sum_{k=0}^{p} \binom{2n+1}{p} \binom{p}{k} F_k\right)^{m+1} \geq 5^n (m+1)^{m+1} L_{2n+1}.$$

H-761 Proposed by Ovidiu Furdui, Campia Turzii, Romania.
(*Dedicated to the memory of Paul S. Bruckman*)

Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \cdots\right)^2 = \frac{\pi^2 \ln 2}{6} - \frac{\ln^3 2}{3} - \frac{3}{4}\zeta(3),$$

where ζ denotes the Riemann zeta function.

H-762 Proposed by George Hisert, Berkeley, California.

Prove that for any positive integers r and n and positive integer p,

(i) $$\sum_{k=0}^{\lfloor (p-1)/2 \rfloor} (-1)^k \binom{p}{k} F_{2(p-2k)r} (F_{n+4r}^{p-k} F_n^k - (-1)^p F_{n+4r}^k F_n^{p-k}) = F_{4r}^p F_{p(n+2r)};$$

(ii) $$\sum_{k=0}^{\lfloor (p-1)/2 \rfloor} (-1)^k \binom{p}{k} F_{2(p-2k)r} (L_{n+4r}^{p-k} L_n^k - (-1)^p L_{n+4r}^k L_n^{p-k}) = F_{4r}^p L_{p(n+2r)}.$$

H-763 **Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania.**

Prove that:

(i) $\displaystyle\sum_{k=1}^{n} \frac{F_k^4}{k^2} \geq \frac{6F_n^2 F_{n+1}^2}{n(n+2)(2n+1)}$;

(ii) $\displaystyle\sum_{k=1}^{n} \frac{F_k^6}{k^2} \geq \frac{4F_n^3 F_{n+1}^3}{n^2(n+1)^2}$;

(iii) $\displaystyle\sum_{k=1}^{n} \frac{F_k^6}{k^4} \geq \frac{36F_n^3 F_{n+1}^3}{n^2(n+1)^2(2n+1)^2}$;

(iv) $\displaystyle\sum_{k=1}^{n} \frac{F_k^8}{k^3} \geq \frac{4F_n^4 F_{n+1}^4}{n^2(n+1)^2}$;

(v) $\displaystyle\sum_{k=1}^{n} \frac{F_k^4}{k^3} \geq \frac{4F_n^2 F_{n+1}^2}{n^2(n+1)^2}$;

(vi) $\displaystyle\sum_{k=1}^{n} \frac{F_k^6}{k^6} \geq \frac{16F_n^3 F_{n+1}^3}{n^4(n+1)^4}$.

H-764 **Proposed by H. Ohtsuka, Saitama, Japan.**

Let $\binom{n}{k}_F$ denote the Fibonomial coefficient. For $n \geq 1$, prove that

(i) $\displaystyle\sum_{k=0}^{n} F_{2(n-k)} \binom{2n}{k}_F = \frac{F_n F_{n+1}}{F_{2n-1}} \binom{2n}{n}_F$;

(ii) $\displaystyle\sum_{k=0}^{n} F_{2(n-k)} \binom{2n}{k}_F^2 = \frac{F_n}{L_n} \binom{2n}{n}_F^2$.

SOLUTIONS

Asymptotic Approximation of a Function Defined by a Sum

H-731 **Proposed by Anastasios Kotronis, Athens, Greece.**
(Vol. 51, No. 1, February 2013) Show that

$$f(x) := \sum_{n=1}^{\infty} \frac{n\cosh(nx)}{\sinh(n\pi)} = \frac{1}{(\pi-x)^2} + \frac{3\pi - 12}{12\pi} + O\left((\pi - x)\right) \quad \text{as} \quad x \to \pi^-.$$

Solution by the proposer.

At first we prove a trivial lemma.

Lemma 1. *Let $a_n = a_1 + (n-1)a$ and $b_n = b_1 b^{n-1}$ with $a, a_1, b_1 \in \mathbb{R}$, $b \neq 1$ be an arithmetic and a geometric progression, respectively. If $c_n := a_n b_n$, then*

$$\sum_{k=1}^{n} c_k = \frac{a_1 b_1 (1 - b^n)}{1 - b} + \frac{a b_1 b}{(1-b)^2}(1 - nb^{n-1} + (n-1)b^n).$$

Proof. We have

$$\begin{aligned}\sum_{k=1}^{n} c_n &= \sum_{k=1}^{n}(a_1+(k-1)a)b_1 b^{k-1}\\ &= a_1 b_1 \sum_{k=1}^{n} b^{k-1} + ab_1 b \sum_{k=1}^{n-1} kb^{k-1}\\ &= \frac{a_1 b_1(1-b^n)}{1-b} + ab_1 b \frac{d\left(\sum_{k=1}^{n-1} b^k\right)}{db}\\ &= \frac{a_1 b_1(1-b^n)}{1-b} + ab_1 b \frac{d\left(\frac{b-b^n}{1-b}\right)}{db}\\ &= \frac{a_1 b_1(1-b^n)}{1-b} + \frac{ab_1 b}{(1-b)^2}(1-nb^{n-1}+(n-1)b^n).\end{aligned}$$

□

For $x \to \pi^-$, we have

$$\begin{aligned}f(x) &= \sum_{n=1}^{+\infty} ne^{-n(\pi-x)}\frac{1+e^{-2nx}}{1-e^{-2n\pi}}\\ &= \sum_{n=1}^{+\infty}\left(ne^{-n(\pi-x)}\left(1+e^{-2nx}\right)\sum_{k=0}^{+\infty} e^{-2nk\pi}\right)\\ &= \sum_{n=1}^{+\infty}\left(\left(ne^{-n(\pi-x)}+ne^{-n(\pi+x)}\right)\sum_{k=0}^{+\infty} e^{-2nk\pi}\right)\\ &= \sum_{n=1}^{+\infty}\left(\sum_{k=0}^{+\infty} ne^{-n((2k+1)\pi-x)}+\sum_{k=0}^{+\infty} ne^{-n((2k+1)\pi+x)}\right)\\ &= \sum_{k=0}^{+\infty}\sum_{n=1}^{+\infty} ne^{-n((2k+1)\pi-x)}+\sum_{k=0}^{+\infty}\sum_{n=1}^{+\infty} ne^{-n((2k+1)\pi+x)}\\ &= \frac{1}{4}\left(\sum_{k=0}^{+\infty}\operatorname{csch}^2\left(\frac{(2k+1)\pi-x}{2}\right)+\sum_{k=0}^{+\infty}\operatorname{csch}^2\left(\frac{(2k+1)\pi+x}{2}\right)\right)\\ &= \frac{1}{4}\left(\operatorname{csch}^2\left(\frac{\pi-x}{2}\right)+\sum_{k=1}^{+\infty}\operatorname{csch}^2\left(\frac{(2k+1)\pi-x}{2}\right)+\sum_{k=1}^{+\infty}\operatorname{csch}^2\left(\frac{(2k-1)\pi+x}{2}\right)\right). \quad (1)\end{aligned}$$

In the above arguments, we used Lemma 1 with $a = a_1 = 1$ and $b = b_1 = e^{-((2k+1)\pi\pm x)}$.
Now, as $x \to \pi^-$, we have

$$\begin{aligned}\operatorname{csch}^2\left(\frac{\pi-x}{2}\right)&=\frac{4}{\left(e^{\frac{\pi-x}{2}}-e^{-\frac{\pi-x}{2}}\right)^2}\\&=\frac{4}{\left(\pi-x+\frac{(\pi-x)^3}{24}+\mathcal{O}\left((\pi-x)^5\right)\right)^2}\\&=\frac{4}{(\pi-x)^2}\left(1+\frac{(\pi-x)^2}{24}+\mathcal{O}\left((\pi-x)^4\right)\right)^{-2}\\&=\frac{4}{(\pi-x)^2}-\frac{1}{3}+\mathcal{O}\left((\pi-x)^2\right),\end{aligned}\tag{2}$$

and for $k\geq 1$, $x\to\pi^-$

$$\begin{aligned}\operatorname{csch}^2\left(\frac{(2k+1)\pi-x}{2}\right)&=\left(\frac{2}{\left(e^{k\pi}-e^{-k\pi}\right)\left(1+\mathcal{O}(\pi-x)\right)}\right)^2\\&=\operatorname{csch}^2(k\pi)+\mathcal{O}\left((\pi-x)\operatorname{csch}^2(k\pi)\right),\end{aligned}\tag{3}$$

and similarly

$$\operatorname{csch}^2\left(\frac{(2k-1)\pi+x}{2}\right)=\operatorname{csch}^2(k\pi)+\mathcal{O}\left((\pi-x)\operatorname{csch}^2(k\pi)\right).\tag{4}$$

Now with the aid of (4), (3), (2), (1), we get

$$\begin{aligned}f(x)&=\frac{1}{(\pi-x)^2}-\frac{1}{12}+\frac{1}{2}\sum_{k=1}^{+\infty}\operatorname{csch}^2(k\pi)+\mathcal{O}(\pi-x)\\&=\frac{1}{(\pi-x)^2}-\frac{1}{4\pi}+\mathcal{O}(\pi-x)\qquad(x\to\pi^-).\end{aligned}$$

For the result

$$\sum_{k=1}^{+\infty}\operatorname{csch}^2(k\pi)=\frac{\pi-3}{6\pi},$$

which was used above, we refer the reader to [1] for a solution via complex analysis methods.

References

[1] R. E. Shafer, *Problem 5063, with solutions by A. E. Livingston and J. Raleigh*, Amer. Math. Monthly, **70** (1963), 1110–1111.

Errata: Note that the second term in the expansion is $-\frac{1}{4\pi}$ instead of $\frac{3\pi-12}{12\pi}$. This is due to a miscalculation in the original submission.

Also solved by Paul S. Bruckman.

Some Properties of Catalan Numbers

H-732 Proposed by N. Gauthier, Kingston, ON (Vol. 51, No. 1, February 2013)

In the following, C_k is the kth Catalan number with the convention that $C_k=0$ if $k<0$.

(1) For nonnegative integers m, n let

$$c_m(n) = \sum_{k\geq 0} (-1)^k \binom{n-k}{k} C_{n-m-k}.$$

Find a closed form for $c_m(n)$.

(2) For nonnegative integers m, n let

$$G_m(n) = \sum_{k\geq 0} (-1)^k \binom{n-k}{k} \binom{2(n-m-k)}{n-m-k}.$$

(a) Show that $G_m(n) = 0$ for $0 \leq n \leq m-1$.
(b) Find a closed form for $G_m(n)$ if $n \geq 2m$.
(c) Show that $G_m(n+m)$ is a polynomial of degree n in m and express the polynomial coefficients as a ratio of two determinants.

Solution by Harris Kwong, SUNY Fredonia, Fredonia, NY.

(1) Recall that the generating function for the Catalan numbers is

$$C(x) = \sum_{n\geq 0} C_n x^n = \frac{1-\sqrt{1-4x}}{2x},$$

so that

$$A(x) = \sum_{n\geq 0} C_{n-m} x^n = x^m \sum_{n\geq m} C_{n-m} x^{n-m} = x^m C(x).$$

Next, we derive the generating function for $c_m(n)$:

$$c_m(x) = \sum_{n\geq 0} c_m(n) x^n = \sum_{n\geq 0} \left[\sum_{k\geq 0} (-1)^k \binom{n-k}{k} C_{n-m-k} \right] x^n.$$

Set $n = k + \ell$ so that we can write

$$c_m(x) = \sum_{\ell\geq 0} C_{\ell-m} x^\ell \sum_{k=0}^{\ell} \binom{\ell}{k} (-x)^k = \sum_{\ell\geq 0} C_{\ell-m} [x(1-x)]^\ell = A\big(x-x^2\big).$$

Since

$$C\big(x-x^2\big) = \frac{1-\sqrt{1-4\big(x-x^2\big)}}{2x(1-x)} = \frac{1-(1-2x)}{2x(1-x)} = \frac{1}{1-x},$$

we find $c_m(x) = x^m(1-x)^{m-1}$. Therefore, by comparing coefficients of x^n, we determine that

$$c_n(m) = (-1)^{n-m} \binom{m-1}{n-m}.$$

(2) By now, it is obvious that the same argument would lead to

$$G_m(x) = \sum_{n\geq 0} G_m(n) x^n = \big(x-x^2\big)^m B\big(x-x^2\big),$$

where

$$B(x) = \sum_{n\geq 0} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}.$$

Since $B\big(x-x^2\big)=\frac{1}{1-2x}$, we obtain $G_m(x)=\frac{x^m(1-x)^m}{1-2x}$, which immediately proves (a). After expansion, and using convolution, we find

$$G_m(x)=x^m\left[\sum_{i=0}^{m}(-1)^i\binom{m}{i}x^i\right]\left[\sum_{\ell\geq 0}2^\ell x^\ell\right]=\sum_{k\geq 0}\left[\sum_{i=0}^{\min(m,k)}(-1)^i\binom{m}{i}2^{k-i}\right]x^{m+k}.$$

Therefore,

$$G_m(n)=\sum_{i=0}^{\min(m,n-m)}(-1)^i\binom{m}{i}2^{n-m-i}.$$

In particular, for $n\geq 2m$,

$$G_m(n)=2^{n-m}\sum_{i=0}^{m}(-1)\binom{m}{i}2^{m-i}=2^{n-m}\cdot(2-1)^m=2^{n-m},$$

which settles (b). To prove (c), we note that for $0\leq n<m$,

$$G_m(n+m)=\sum_{i=0}^{n}(-1)^i\binom{m}{i}2^{n-i},$$

which is clearly a polynomial of degree n in m. Let $G_m(n+m)=\sum_{i=0}^{n}a_im^i$. By setting $m=0,1,2,\ldots,n-1$ in the two expressions for $G_m(n+m)$, we obtain a linear system of $n+1$ equations and $n+1$ unknowns. For example, the linear system for $n=3$ is

$$\begin{array}{lllllllll}
a_0 & & & & & & & = & 8,\\
a_0 & + & a_1 & & & & & = & 4,\\
a_0 & + & 2a_1 & + & 4a_2 & & & = & 2,\\
a_0 & + & 3a_1 & + & 9a_2 & + & 27a_3 & = & 1.
\end{array}$$

According to Cramer's Rule, we can express each a_i as a ratio of two determinants. The proof is now complete.

Also solved by Paul S. Bruckman and the proposer.

On a Complex Sequence

H-733 Proposed by H. Ohtsuka, Saitama, Japan.
(Vol. 51, No. 1, February 2013)

Define the sequence $\{H_n\}_{n\geq -1}$ given by $H_{-1}=\mathbf{i}$, $H_0=0$, $H_{n+2}=H_{n+1}-\mathbf{i}H_n$ for $n\geq -1$, where $\mathbf{i}=\sqrt{-1}$. Find an explicit formula for $\sum_{k=1}^{n}H_k^4$.

Solution by Harris Kwong, SUNY Fredonia, Fredonia, NY.

Since $H_k=0$, we could start the summation with $k=0$. From Binet's formula

$$H_k=\frac{\alpha^k-\beta^k}{\alpha-\beta},$$

where $\alpha=\frac{1+\sqrt{1-4\mathbf{i}}}{2}$, and $\beta=\frac{1-\sqrt{1-4\mathbf{i}}}{2}$, we find

$$(\alpha-\beta)^4H_k^4=4\alpha^{4k}-4\alpha^{3k}\beta^k+6\alpha^{2k}\beta^{2k}-4\alpha^k\beta^{3k}+\beta^{4k}.$$

Since $\alpha\beta = \mathbf{i}$, we find $\beta^2(1-\alpha^4) = \beta^2 + \alpha^2$. Thus,

$$\sum_{k=0}^{n} \alpha^{4k} = \frac{1-\alpha^{4n+4}}{1-\alpha^4} = \frac{\beta^2(1-\alpha^{4n+4})}{\alpha^2+\beta^2} = \frac{\beta^2+\alpha^{4n+2}}{\alpha^2+\beta^2}.$$

A similar result holds for the summation of β^{4k}. We deduce that

$$\sum_{k=0}^{n} (\alpha^{4k} + \beta^{4k}) = 1 + \frac{a^{4n+2}+\beta^{4n+2}}{\alpha^2+\beta^2}.$$

In a similar manner, we find $\beta(1-\alpha^3\beta) = \beta + \alpha = 1$. Hence,

$$\sum_{k=0}^{n} \alpha^{3k}\beta^k = \frac{1-\alpha^{3n+3}\beta^{n+1}}{1-\alpha^3\beta} = \beta(1-\alpha^{3n+3}\beta^{n+1}) = \beta + \mathbf{i}^n\alpha^{2n+1},$$

which leads to

$$\sum_{k=0}^{n} (\alpha^{3k}\beta^k + \alpha^k\beta^{3k}) = 1 + \mathbf{i}^n(\alpha^{2n+1} + \beta^{2n+1}).$$

Finally, we have

$$\sum_{k=0}^{n} \alpha^{2k}\beta^{2k} = \frac{1-(\alpha\beta)^{2n+2}}{1-(\alpha\beta)^2} = \frac{1+(-1)^n}{2}.$$

Combining these results, along with $\alpha - \beta = \sqrt{1-4\mathbf{i}}$, and $\alpha^2 + \beta^2 = 1 - 2\mathbf{i}$, we determine that

$$\sum_{k=0}^{n} H_k^4 = \frac{1}{(1-4\mathbf{i})^2}\left[\frac{\alpha^{4n+2}+\beta^{4n+2}}{1-2\mathbf{i}} - 4\mathbf{i}^n(\alpha^{2n+1}+\beta^{2n+1}) + 3(-2)^n\right].$$

We can simplify the formula by introducing the associated Lucas-type sequence K_n defined by $K_{-1} = -\mathbf{i}$, $K_0 = 2$, and $K_{n+2} = K_{n+1} - \mathbf{i}K_n$ for $n \geq -1$. Then $K_n = \alpha^n + \beta^n$, so that

$$\sum_{k=0}^{n} H_k^4 = \frac{1}{(1-4\mathbf{i})^2}\left[\frac{1}{1-2\mathbf{i}}\,K_{4n+2} - 4\mathbf{i}^n K_{2n+1} + 3(-2)^n\right].$$

We can also express the formula in terms of H_n, with the help of the formulas

$$K_{4n+2} = (\alpha^{2n+1} - \beta^{2n+1})^2 + 2(\alpha\beta)^{2n+1} = (1-4\mathbf{i})H_{2n+1}^2 + 2(-1)^n\mathbf{i},$$
$$K_{2n+1} = K_{2n+2} + \mathbf{i}K_{2n} = (1-4\mathbf{i})(H_{n+1}^2 + \mathbf{i}H_n^2) + 4\mathbf{i}^{n+1}.$$

However, the result is somewhat more complicated, hence is omitted here.

Also solved by Paul S. Bruckman, G. C. Greubel, Zbigniew Jakubczyk, and the proposer.