ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, IMATE, UNAM, AP. POSTAL 61-3 (XANGARI), CP 58 089, MORELIA, MICHOACAN, MEXICO, or by e-mail at flucamatmor.unam.mx as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-681 Proposed by N. Gauthier, Kingston, ON

For a real variable $z \neq 0$ consider the sets of generalized Fibonacci and Lucas polynomials, $\{f_n = f_n(z) : n \in \mathbb{Z}\}$ and $\{l_n = l_n(z) : n \in \mathbb{Z}\}$, given by the recurrences

$$f_{n+2} = zf_{n+1} + f_n, \qquad \text{and} \qquad l_{n+2} = zl_{n+1} + l_n, \qquad \text{for all} \quad n \in \mathbb{Z},$$

with $f_0 = 0,\ f_1 = 1,\ l_0 = 2,\ l_1 = z$. Note that $f_{-n} = (-1)^{n+1} f_n$ and $l_{-n} = (-1)^n l_n$. Let r be a nonnegative integer and $p,\ q$ be positive integers.

(a) Prove that

$$\sum_{k\geq 0} (-1)^k k \binom{r}{k} f_p^k f_{p+q}^{r-k} l_{qk} = (-1)^{q+1} r f_p f_q^{r-1} l_{pr-(p+q)}.$$

(b) Find a general formula for $\sum_{k\geq 0} (-1)^k k^m \binom{r}{k} f_p^k f_{p+q}^{r-k} l_{qk}$ for any nonnegative integer m.

H-682 Proposed by G. C. Greubel, Newport News, VA

Given the generalized Laguerre polynomials,

$$L_n^{(\alpha,\beta,\gamma)}(x) = \frac{(\alpha+1)_n}{n!}\, {}_2F_2(-n, \beta+1; \alpha+1, \gamma+1; x),$$

show that

$$L_n^{(\alpha,\beta,\gamma)}\left(x^2\right) = \frac{(\alpha+1)_n}{n!} \sum_{r=0}^{n} \frac{(-n)_r (\beta+1)_r}{r!(\alpha+1)_r(\gamma+1)_r} \cdot \theta_r^n \cdot F_{2r+1}(x),$$

where $F_n(x)$ is the Fibonacci polynomial and θ_r^n is a hypergeometric polynomial of type ${}_3F_3$.

H-683 Proposed by Guodong Liu, Huizhou, China

The generalized binomial coefficients of the first kind $\sigma_k(x_1, x_2, \ldots, x_n)$ and the generalized binomial coefficients of the second kind $\tau_k(x_1, x_2, \ldots, x_n)$ are defined by

$$(1-x_1x)(1-x_2x)\cdots(1-x_nx)=\sum_{k=0}^{\infty}\sigma_k(x_1,x_2,\ldots,x_n)x^k$$

and

$$\frac{1}{(1-x_1x)(1-x_2x)\cdots(1-x_nx)}=\sum_{k=0}^{\infty}\tau_k(x_1,x_2,\ldots,x_n)x^k,$$

respectively. For any positive integers n and k prove that

$$x_1^k+x_2^k+\cdots+x_n^k=-\sum_{j=1}^{k}j\sigma_j(x_1,\ldots,x_n)\tau_{k-j}(x_1,\ldots,x_n).$$

H-684 Proposed by N. Gauthier, Kingston, ON

For an arbitrary positive integer N and a real number $a>2$, consider the following $N\times N$ matrix:

$$\mathbf{A}=\begin{pmatrix} a & 1 & 0 & \ldots & 0 & 0\\ 1 & a & 1 & \ldots & 0 & 0\\ \ldots & \ldots & \ldots & \ldots & \ldots & 0\\ 0 & 0 & 0 & \ldots & 1 & 0\\ 0 & 0 & 0 & \ldots & a & 1\\ 0 & 0 & 0 & \ldots & 1 & a\end{pmatrix}.$$

(a) Find a closed form expression for the determinant of **A**.
(b) Find the eigenvalues of **A** and show that they are real, positive, and distinct.
(c) Find expressions for the eigenvectors of **A**.

SOLUTIONS

Some Fibonacci Limits

H-664 Proposed by A. Cusumano, Great Neck, NY (Vol. 45, No. 4, November 2007)

If a is a positive integer, prove that

$$\lim_{n\to\infty}(F_n^{1/a}+F_{n+a}^{1/a}-F_{n+2a}^{1/a})=0 \quad\text{and}\quad \lim_{n\to\infty}\frac{F_n^{1/a}+F_{n+a}^{1/a}}{F_{n+2a}^{1/a}}=1.$$

Based on a solution by G. C. Greubel, Newport News, VA

From a definition of the Fibonacci numbers, we have

$$\left(\sqrt{5}F_n\right)^{1/a}=\alpha^{n/a}\left(1-(-1)^n\beta^{2n}\right)^{1/a}=\alpha^{n/a}\left(1+O(\alpha^{-2n})\right)=\alpha^{n/a}+O(\alpha^{-(2-1/a)n}).$$

Let

$$(\sqrt{5})^{1/a}\sigma_n=\left(\sqrt{5}F_n\right)^{1/a}+\left(\sqrt{5}F_{n+a}\right)^{1/a}-\left(\sqrt{5}F_{n+2a}\right)^{1/a}.$$

We then have

$$\begin{aligned}(\sqrt{5})^{1/a}\sigma_n &= \alpha^{n/a}\left(1-(-1)^n\beta^{2n}\right)^{1/a}+\alpha^{(n+a)/a}\left(1-(-1)^{n+a}\beta^{2n+2a}\right)^{1/a}\\ &\quad -\alpha^{(n+2a)/a}\left(1-(-1)^n\beta^{2n+4a}\right)^{1/a}\\ &= \alpha^{n/a}+O(\alpha^{-(2-1/a)n})+\alpha^{n/a+1}+O(\alpha^{-(2-1/a)n})-\alpha^{n/a+2}+O(\alpha^{-(2-1/a)n})\\ &= \alpha^{n/a}(1+\alpha-\alpha^2)+O(\alpha^{-(2-1/a)n})=O(\alpha^{-(2-1/a)n}),\end{aligned}$$

so

$$\lim_{n\to\infty}(\sqrt{5})^{1/a}\sigma_n=0,$$

where we used

$$\lim_{n\to\infty}\alpha^{-(2-1/a)n}=0. \tag{1}$$

For the second limit, let

$$\phi_n=\frac{\left(\sqrt{5}F_n\right)^{1/a}+\left(\sqrt{5}F_{n+a}\right)^{1/a}}{\left(\sqrt{5}F_{n+2a}\right)^{1/a}}.$$

By a similar method, we have

$$\begin{aligned}\phi_n &= \frac{\alpha^{n/a}\left(1-(-1)^n\beta^{2n}\right)^{1/a}+\alpha^{(n+a)/a}\left(1-(-1)^{n+a}\beta^{2n+2a}\right)^{1/a}}{\alpha^{(n+2a)/a}\left(1-(-1)^n\beta^{2n+4a}\right)^{1/a}}\\ &= \frac{\alpha^{n/a}+O(\alpha^{-(2-1/a)n})+\alpha^{n/a+1}+O(\alpha^{-(2-1/a)n})}{\alpha^{n/a+2}+O(\alpha^{-(2-1/a)n})}\\ &= \frac{1+\alpha+O(\alpha^{-2n})}{\alpha^2+O(\alpha^{-2n})}=1+O(\alpha^{-2n}),\end{aligned}$$

and using (1) we conclude that the last expression above tends to 1 when n tends to infinity. The desired limits now follow from the above estimates.

Also solved by Paul S.Bruckman.

<u>The Bilateral Binomial Theorem and Fibonacci Numbers</u>

<u>H-665</u> Proposed by G. C. Greubel, Newport News, VA (Vol. 46, No. 1, February 2008)

Given the bilateral series

$${}_1H_1(a;b;x)=\sum_{n=-\infty}^{n=\infty}\frac{(a)_n}{(b)_n}x^n$$

derive general expressions that reduce to the equations

$$\sum_{r=0}^{1}\sum_{n=0}^{\infty}\binom{2m}{2n}{}_1H_1(2n-2m;2n+1;(-1)^r\sqrt{5})=5\cdot 4^{m-1}L_{2m}$$

$$\sum_{r=0}^{1}\sum_{n=0}^{\infty}(-1)^r\binom{2m}{2n}{}_1H_1(2n-2m;2n+1;(-1)^{r+1}\sqrt{5})=5^{3/2}\cdot 4^{m-1}F_{2m}.$$

Solution by Paul S. Bruckman, Sointula, Canada

The solution depends on what has been called Horn's Theorem or the Bilateral Binomial Theorem (BBT for short). This theorem is due to Martin Erick Horn. Other proofs of it are known including a proof by the proposer of this problem. Horn first proposed it as a problem on July 24, 2003 (see [1]).

BBT states the following, where x is real, $y \geq 0$ is an integer and z is complex:

$$(1+z)^x = z^y \binom{x}{y} {}_1H_1(y-x; y+1; -z). \tag{1}$$

Recall that the definition of the Pochhammer symbol $(a)_n$ when $n < 0$ is

$$(a)_{-n} = \frac{(-1)^n}{(1-a)_{-n}} \qquad \text{and is valid for} \quad n = 1, 2, \ldots, \quad \text{and} \quad a \neq 1, 2, \ldots. \tag{2}$$

In (1), we set $x = 2m,\ y = 2n,\ z = \pm\sqrt{5}$. This yields the following:

$$(1+\sqrt{5})^m + (1-\sqrt{5})^m = 5^n \binom{2m}{2n} \left({}_1H_1(2n-2m; 2n+1; \sqrt{5}) + {}_1H_1(2n-2m; 2n+1; -\sqrt{5}\right),$$

or

$$2^{2m} L_{2m} = 5^n \binom{2m}{2n} \left({}_1H_1(2n-2m; 2n+1; \sqrt{5}) + {}_1H_1(2n-2m; 2n+1; -\sqrt{5}\right). \tag{3}$$

Likewise,

$$(1+\sqrt{5})^m - (1-\sqrt{5})^m = 5^n \binom{2m}{2n} \left({}_1H_1(2n-2m; 2n+1; \sqrt{5}) - {}_1H_1(2n-2m; 2n+1; -\sqrt{5}\right),$$

or

$$2^{2m} F_{2m}\sqrt{5} = 5^n \binom{2m}{2n} \left({}_1H_1(2n-2m; 2n+1; \sqrt{5}) - {}_1H_1(2n-2m; 2n+1; -\sqrt{5}\right). \tag{4}$$

Next, we divide both sides of (3) and (4) by 5^n and we sum the resulting relations over all $n \geq 0$. Notice that the apparently infinite sums on the right are actually finite since the binomial coefficients $\binom{2m}{2n}$ are zero for $n \geq m+1$. Thus, we may sum over all $n \geq 0$ in the left hand side but only over $n \in [0, m]$ in the right hand side. Moreover, we see that the bilateral series appearing in each of such sums are well-defined. That is, the denominators of $(a)_{-j}$ and $(b)_{-j}$ are nonzero for $j > 0$ and $0 \leq n \leq m$.

Therefore, we see that the first sum in the statement of the problem is equal to

$$2^{2m} L_{2m} \sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n = 4^m L_{2m} \left(\frac{5}{4}\right) = 5 \cdot 4^{m-1} L_{2m}.$$

Similarly, the second sum in the statement of the problem is equal to

$$2^{2m} F_{2m}\sqrt{5} \sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n = 4^m F_{2m}\sqrt{5} \left(\frac{5}{4}\right) = 5^{3/2} \cdot 4^{m-1} F_{2m}.$$

[1] M. E. Horn, *Problem 001- 03*, `http://www.siam.org/journals/problems/03-001.htm.`

Also solved by the proposer.

Some Identities Involving Pell Numbers and Binomial Coefficients

H-666 Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 46, No. 1, February 2008)

The Pell and Pell-Lucas numbers are defined by $P_0 = 0,\ P_1 = 1,\ Q_0 = 2,\ Q_1 = 2$ and $P_{n+1} = 2P_n + P_{n-1},\ Q_{n+1} = 2Q_n + Q_{n-1}$ for all $n \geq 1$. Prove that, for all positive integers n, we have

$$P_{2n-1} = 2^{n-2}(4^{n-1}+1) - 2^{2-n}\sum_{k=0}^{\lfloor (n-3)/4\rfloor}\binom{4n-2}{2n-8k-5},$$

$$Q_{2n} = 2^{n}(2^{2n-1}+1) - 2^{3-n}\sum_{k=0}^{\lfloor (n-2)/4\rfloor}\binom{4n}{2n-8k-4}.$$

Solution by G. C. Greubel, Newport News, VA [1]

Consider the series

$$T_m = \sum_{k=0}^{a}\binom{2m}{m-8k-4}, \tag{1}$$

where $a = \lfloor \frac{m-4}{8} \rfloor$ and its subsum

$$V_m(x) = \sum_{k=0}^{m}\binom{2m}{m-k}x^k$$

for $x = 1$ corresponding to the summation terms congruent to 4 modulo 8. It can be seen that

$$V_m(x) = \sum_{k=0}^{m}\binom{2m}{m-k}x^k = \sum_{k=0}^{m}\binom{2m}{k}x^{m-k} = \sum_{k=m}^{2m}\binom{2m}{k}x^{k-m}.$$

Now consider the combination $V_m(x) + V_m(x^{-1})$ which is given by

$$\begin{aligned} V_m(x) + V_m(x^{-1}) &= \sum_{k=0}^{m}\binom{2m}{k}x^{m-k} + \sum_{k=0}^{m}\binom{2m}{k}x^{k-m} = \sum_{k=0}^{m}\binom{2m}{k}x^{m-k} + \sum_{k=m}^{2m}\binom{2m}{k}x^{m-k} \\ &= x^m\sum_{k=0}^{2m}\binom{2m}{k}x^{-k} + \binom{2m}{m} = \left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)^{2m} + \binom{2m}{m}. \end{aligned}$$

Using the identity

$$\sum_{r=0}^{7} e^{\frac{n\pi i}{4}} = \begin{cases} 0 & \text{if } 4 \nmid n \\ 8 & \text{if } 4 \mid n \end{cases},$$

we get

$$T_m = \frac{1}{8}\sum_{k=0}^{m}\binom{2m}{m-k}\cdot\sum_{r=0}^{7} e^{\frac{\pi r(k-4)}{4}} = \frac{1}{8}\sum_{r=0}^{7}(-1)^r\cdot\sum_{k=0}^{m}\binom{2m}{m-k}e^{\frac{rk\pi i}{4}} = \frac{1}{8}\sum_{r=0}^{7}(-1)^r V_m\left(e^{\frac{r\pi i}{4}}\right). \tag{2}$$

[1] This solution follows the same line as that of Paul S. Bruckman's solution to problem H-651.

Using the above formula with r replaced by $-r$ modulo 8, we have

$$T_m = \frac{1}{8}\sum_{r=0}^{7}(-1)^r V_m\left(e^{-\frac{r\pi i}{4}}\right). \tag{3}$$

Adding equations (2) and (3) we get

$$\begin{aligned} T_m &= \frac{1}{16}\sum_{r=0}^{7}(-1)^r\left[V_m\left(e^{\frac{r\pi i}{4}}\right)+V_m\left(e^{-\frac{r\pi i}{4}}\right)\right] = \frac{1}{16}\sum_{r=0}^{7}(-1)^r\left[\left(e^{\frac{r\pi i}{8}}+e^{-\frac{r\pi i}{8}}\right)^{2m}+\binom{2m}{m}\right]\\ &= \frac{1}{16}\sum_{r=0}^{7}(-1)^r\left[4^m\left(\cos\frac{\pi r}{8}\right)^{2m}+\binom{2m}{m}\right]. \end{aligned}$$

Since $\sum_{r=0}^{7}(-1)^r = 0$, the above terms involving $\binom{2m}{m}$ cancel and we get

$$T_m = 4^{m-2}\sum_{r=0}^{7}(-1)^r\left(\cos\frac{\pi r}{8}\right)^{2m}.$$

Now recall that

$$\begin{aligned} \cos\frac{\pi}{8} &= \frac{\sqrt{2+\sqrt{2}}}{2}, \qquad \cos\frac{2\pi}{8}=\frac{1}{\sqrt{2}}, \qquad \cos\frac{3\pi}{8}=\frac{\sqrt{2-\sqrt{2}}}{2}, \qquad \cos\frac{4\pi}{8}=0,\\ \cos\frac{5\pi}{8} &= -\frac{\sqrt{2-\sqrt{2}}}{2}, \qquad \cos\frac{6\pi}{8}=-\frac{1}{\sqrt{2}}, \qquad \cos\frac{7\pi}{8}=-\frac{\sqrt{2+\sqrt{2}}}{2}, \end{aligned}$$

so

$$T_m = 4^{m-2}\left(1-\frac{1}{2^{m-1}}-\frac{1}{2^{3m/2-1}}[(1+\sqrt{2})^m+(-1)^m(1-\sqrt{2})^m]\right),$$

or

$$(1+\sqrt{2})^m+(-1)^m(1-\sqrt{2})^m = 2^{3m/2-1}+2^{m/2}-2^{3-m/2}T_m.$$

When $m = 2n-1$ is odd, then the last relation above becomes

$$(1+\sqrt{2})^{2n-1}-(1-\sqrt{2})^{2n-1} = 2^{3n-2-1/2}+2^{n-1/2}-2^{3-n+1/2}T_{2n-1}.$$

Using the known formula for the Pell number

$$P_m = \frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^m-(1-\sqrt{2})^m\right], \qquad \text{for all} \quad m = 0,\ 1,\ \ldots,$$

with $m = 2n-1$, we get

$$P_{2n-1} = 2^{n-2}\left(4^{n-1}+1\right)-2^{2-n}T_{2n-1}. \tag{4}$$

When $m = 2n$ is even, then using the formula for the Pell-Lucas number

$$Q_m = (1+\sqrt{2})^m+(1-\sqrt{2})^m, \qquad \text{for all} \quad m = 0,\ 1,\ \ldots,$$

with $m = 2n$, we get

$$Q_{2n} = 2^n\left(2^{2n-1}+1\right)-2^{3-n}T_{2n}. \tag{5}$$

Replacing the values for T_m with the sums of binomial coefficients (1) into relations (4) and (5), we arrive at the desired relations

$$P_{2n-1} = 2^{n-2}(4^{n-1}+1) - 2^{2-n}\sum_{k=0}^{a}\binom{4n-2}{2n-8k-5}, \qquad \text{and}$$

$$Q_{2n} = 2^{n}(2^{2n-1}+1) - 2^{3-n}\sum_{k=0}^{b}\binom{4n}{2n-8k-4},$$

with $a = \lfloor \frac{n-3}{4} \rfloor$ and $b = \lfloor \frac{n-2}{4} \rfloor$.

Also solved by Paul S. Bruckman and the proposer.

π and Lucas sequences

H-667 Proposed by Herman Roelants, Leuven, Belgium (Vol. 46, No. 1, February 2008)

Let $u_n = pu_{n-1} + qu_{n-2}$ for all $n \geq 2$, with $u_0 = 0$, $u_1 = 1$ and p, $q > 0$. Prove that

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n u_{2n+1}}{(2n+1)(p^2+4q)^n} t^{2n+1} \qquad \text{with} \qquad t = \frac{2}{1+\sqrt{\frac{p^2+8q}{p^2+4q}}}.$$

Solution by G. C. Greubel, Newport News, VA

Binet's formula for the sequence $(u_n)_{n\geq 0}$ is $u_n = (r^n - s^n)/(r-s)$, where $r = (p+\theta)/2$ and $s = (p-\theta)/2$, with $\theta = \sqrt{p^2+4q} = r - s$.

Thus, we can cast the series

$$S = \sum_{n=0}^{\infty} \frac{(-1)^n u_{2n+1} t^{2n+1}}{(2n+1)(p^2+4q)^n} \tag{1}$$

as

$$S = \sum_{n=0}^{\infty} \frac{(-1)^n \left(r^{2n+1} - s^{2n+1}\right) t^{2n+1}}{(2n+1)\theta^{2n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(\frac{rt}{\theta}\right)^{2n+1} - \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(\frac{st}{\theta}\right)^{2n+1},$$

so

$$S = \tan^{-1}\left(\frac{rt}{\theta}\right) - \tan^{-1}\left(\frac{st}{\theta}\right). \tag{2}$$

By using the known relation $\tan^{-1}(x) - \tan^{-1}(y) = \tan^{-1}((x-y)/(1+xy))$, the formula (2) for S can be written as

$$S = \tan^{-1}\left(\frac{\theta^2 t}{\theta^2 - qt^2}\right). \tag{3}$$

In order to simplify this result, consider the following. Let $\sigma = \delta/\theta$, where $\delta = \sqrt{p^2+8q}$. With these notations, we have $t = 2/(1+\sigma) = 2\theta/(\theta+\delta)$, and

$$\frac{\theta^2 t}{\theta^2 - qt^2} = \frac{2\theta^3(\theta+\delta)}{\theta^2(\theta+\delta)^2 - 4q\theta^2} = \frac{2\theta(\theta+\delta)}{\theta^2 + 2\theta\delta + \delta^2 - 4q} = \frac{2\theta(\theta+\delta)}{2\theta^2 + 2\theta\delta} = 1. \tag{4}$$

Using the calculation (4) in (3), yields $S = \tan^{-1}(1) = \pi/4$, which completes the solution to this problem.

Also solved by Paul S. Bruckman, Kenneth Davenport and the proposer.