ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, IMATE, UNAM, AP. POSTAL 61-3 (XANGARI), CP 58 089, MORELIA, MICHOACAN, MEXICO, or by e-mail at fluca@matmor.unam.mx as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-643 Proposed by John H. Jaroma, Loyola College in Maryland, Baltimore, MD

The *rank of apparition* of a prime p in $\{F_n\}$ is the index of the first term in the Fibonacci sequence that contains p as a divisor. Furthermore, p is said to have *maximal rank of apparition* provided that its rank of apparition in the underlying sequence is either $p-1$ or $p+1$. Recall that a pair of *twin primes* is a pair of consecutive odd integers p and $p+2$ each of which is prime. Determine if both components of a pair of twin primes can simultaneously have maximal rank of apparition in $\{F_n\}$.

H-644 Proposed by José Luis Díaz-Barrero, Barcelona, Spain

Let n be a positive integer. Solve the following system of equations

$$\begin{pmatrix} 1+\frac{1}{F_1} & 1 & \dots & 1 \\ 1 & 1+\frac{1}{F_2} & \dots & 1 \\ \vdots & \vdots & \dots & \vdots \\ 1 & 1 & \dots & 1+\frac{1}{F_n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{pmatrix}.$$

H-645 Proposed by John H. Jaroma, Loyola College in Maryland, Baltimore, MD

(1) Show that every Mersenne prime is a factor of infinitely many L_n.

(2) Show that no Fermat prime is a factor of any L_n.

H-646 Proposed by John H. Jaroma, Loyola College in Maryland, Baltimore, MD

A *Wiefrich prime* is any prime p that satisfies $2^{p-1} \equiv 1 \pmod{p^2}$. Presently, 1093 and 3511 are the only known such primes. Similarly defined is a *Wall-Sun-Sun prime*, which is any prime p such that $F_{p-(5/p)} \equiv 0 \pmod{p^2}$, where $(5/p)$ is the Legendre symbol. There are no known Wall-Sun-Sun primes. More generally, in 1993 P. Montgomery added 23 new solutions to $a^{p-1} \equiv 1 \pmod{p^2}$. This brought to 219 the number of observed solutions to $2 \le a \le 99$ and $3 \le p < 2^{32}$.

Let p be a prime and $n \geq 1$. Prove that there exist infinitely many integers $a > 1$ such that $a^{p-1} \equiv 1 \pmod{p^n}$.

SOLUTIONS

Pisot from Padova

H-629 Proposed by Ernst Herrmann, Siegburg, Germany (Vol. 43, no. 3, August 2005)

Consider the sequence $(a_n)_{n\geq 0}$ of non-negative integers which are defined by $a_0 = a_1 = 0$, $a_2 = 1$ and by the recurrence relation $a_n = a_{n-2} + a_{n-3}$ if $n \geq 3$. Prove that the numbers of the sequence $(a_n)_{n\geq 0}$ can also be defined by the relation

$$-0.5 < a_{n+2} - a_{n+1}^2/a_n \leq 0.5$$

for all sufficiently large n; i.e., for all $n \geq n_0$. Thus, a_{n+2} is uniquely defined if a_n, a_{n+1} and a_{n+2} fulfill the relation. Determine the smallest possible value of n_0.

Based on the solution by the proposer

The characteristic equation of the recurrence is

$$f(x) = x^3 - x - 1.$$

The above polynomial has a real root $\alpha \in (1.3, 1.5)$ and the other roots are complex conjugated, say ρ, $\overline{\rho}$. By looking at the last coefficient, we get that $1 = \alpha|\rho|^2$, therefore $|\rho| = \alpha^{-1/2}$. Using the initial values, one computes that

$$a_n = c_1\alpha^n + c_2\rho^n + c_3\overline{\rho}^n,$$

where c_1, c_2, c_3 are constants. Their numerical values are $c_1 = 0.2344\ldots$, $|c_2| = |c_3| = .4306\ldots$. Hence,

$$u_n = c_1\alpha^n(1 + E_n),$$

where

$$|E_n| \leq \frac{|c_2| + |c_3|}{|c_1|}|\rho|^n\alpha^{-n} < 4\alpha^{-3n/2}, \tag{1}$$

where we used the fact that $|c_2| = |c_3| < 2|c_1|$. Thus,

$$a_{n+2} - \frac{a_{n+1}^2}{a_n} = c_1\alpha^{n+2}\left(1 + E_{n+2} - \frac{(1 + E_{n+1})^2}{1 + E_n}\right). \tag{2}$$

Note that

$$\left|1 + E_{n+2} - \frac{(1 + E_{n+1})^2}{1 + E_n}\right| = \left|\frac{E_n + E_{n+2} + E_nE_{n+2} - 2E_{n+1} - E_{n+1}^2}{1 + E_n}\right|.$$

Since $|\alpha| > 1.3$, we get that if $n > 8$, then $\alpha^{3n/2} > \alpha^{12} > (1.3)^{12} > (1.6)^6 > (2.5)^3 > 8$, so estimate (1) gives $|E_n| < 1/2$. Thus, for $n > 8$, $|1 + E_n| > 1/2$, therefore

$$\left|\frac{E_n + E_{n+2} + E_nE_{n+2} - 2E_{n+1} - E_{n+1}^2}{1 + E_n}\right| \le 2(2|E_n| + |E_{n+2}| + 3|E_{n+1}|) \le 48\alpha^{-3n/2},$$

so, relation (2) gives

$$\left|a_{n+2} - \frac{a_{n+1}^2}{a_n}\right| \le 48|c_1|\alpha|^{n+2}\alpha^{-3n/2} \le \frac{(0.25)\cdot 48}{\alpha^{n/2-2}} = \frac{12}{(1.3)^{n/2-2}},$$

and the right most expression above is smaller than 0.5 if $n > 36$ since for such n we have $\alpha^{n/2-2} > (1.3)^{n/2-2} > (1.3)^{16} > (1.6)^8 > (2.5)^4 > 6^2 > 24$. One can now check by hand, by listing the first 36 values of a_n, that the desired inequality holds in fact starting with $n_0 = 12$.

Also solved by Paul S. Bruckman.

Editor's comment. The recurrence $(a_n)_{n\ge0}$ is related to the Padovan sequence $(P_n)_{n\ge0}$ given by $P_0 = P_1 = P_2 = 1$ and $P_n = P_{n-2} + P_{n-3}$ for all $n \ge 3$. Given positive integers a_0 and a_1 with $a_1 \ge a_0$, let $(a_n)_{n\ge0}$ be the sequence in which a_{n+2} is the closest integer to a_{n+1}^2/a_n for all $n \ge 0$ (if there are two choices for the closest integer, pick one of them). The resulting sequence $(a_n)_{n\ge0}$ is called a *Pisot sequence*. Problem H629 points out that a certain ternary recurrent sequence is a Pisot sequence if $n > n_0$. Conversely, it is not true in general that Pisot sequences satisfy a linear recurrence (of any order) as it was shown by David Boyd.

Fibonacci polynomials and periodic binary recurrences

H-630 Proposed by Mario Catalani, Torino, Italy (Vol. 43, no. 3, August 2005)

Let $F_n(x,y)$ be the bivariate Fibonacci polynomials, defined, for $n \ge 2$, by $F_n(x,y) = xF_{n-1}(x,y) + yF_{n-2}(x,y)$, where $F_0(x,y) = 0$, $F_1(x,y) = 1$. Assume $xy \ne 0$ and $x^2 + 4y \ne 0$.

1. Prove the following identity,

$$x\sum_{k=0}^{n-1}\binom{2n-1-k}{k}(x^2+4y)^{n-k-1}(-y)^k = F_{2n}(x,y).$$

2. Let

$$a_n = \sum_{k=0}^{n-1}\binom{2n-1-k}{k}(-3)^{n-k-1}.$$

Find a recurrence and a closed form for a_n.

Solution by H.-J. Seiffert, Berlin, Germany

Define the sequence of Fibonacci polynomials by $F_0(x) = 0$, $F_1(x) = 1$, and $F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$ for $n \geq 2$. Then (see equations (3.5)–(3.6) and (2.15) in [1]),

$$F_{2n}(x) = \frac{1}{\sqrt{x^2+4}}\Big(\Big(\frac{x+\sqrt{x^2+4}}{2}\Big)^{2n} - \Big(\frac{x-\sqrt{x^2+4}}{2}\Big)^{2n}\Big), \tag{1}$$

and

$$F_{2n}(x) = \sum_{k=0}^{n-1} \binom{2n-1-k}{k} x^{2n-2k-1}. \tag{2}$$

Let $x > 0$. From (1), one finds

$$F_{2n}(i\sqrt{x^2+4}) = \frac{i^{2n-1}}{x}\sqrt{x^2+4}F_{2n}(x), \quad \text{with } i = \sqrt{-1}.$$

Now, (2) with x replaced by $i\sqrt{x^2+4}$ gives

$$F_{2n}(x) = x\sum_{k=0}^{n-1} \binom{2n-1-k}{k} (-1)^k (x^2+4)^{n-k-1}. \tag{3}$$

By analytic continuation, (3) remains valid for all complex x.

1. It is easily verified that $F_n(x,y) = y^{(n-1)/2}F_n(x/y^{1/2})$ for all $n \geq 0$, so that the desired identity follows almost immediately from (3).

2. With $x = i$, (3) implies that $a_n = (-1)^n iF_{2n}(i)$ for all $n \geq 0$. Hence, by (1),

$$a_n = \frac{i}{\sqrt{3}}(-1)^n\Big(\Big(\frac{\sqrt{3}+i}{2}\Big)^{2n} - \Big(\frac{\sqrt{3}-i}{2}\Big)^{2n}\Big), \quad \text{for all } n \geq 0.$$

Using $\cos(\pi/6) = \sqrt{3}/2$, $\sin(\pi/6) = 1/2$, and Euler's relation $e^{it} = \cos t + i\sin t$, one finds

$$a_n = \frac{2}{\sqrt{3}}(-1)^{n-1}\sin\Big(\frac{n\pi}{3}\Big), \quad \text{for all } n \geq 0.$$

Now, the recurrence $a_n = -a_{n-1} - a_{n-2}$ for $n \geq 2$ is easily justified by using the known trigonometric identities.

[1] A. F. Horadam & Bro. J. M. Mahon "Pell and Pell-Lucas polynomials", The Fibonacci Quarterly **23.1** (1985): 7–20.

Also solved by Paul S. Bruckman and the proposer.

A large determinant

H-631 Proposed by Jayantibhai M. Patel, Ahmedabad, India (Vol. 43, no. 4, November 2005)

For any positive integer $n \geq 2$, prove that the value of the following determinant

$$\begin{vmatrix} (F_nF_{n+2}+F_{n+1}^2) & F_n^2 & F_{n+1}^2 & F_{n+2}^2 & -(F_nF_{n+2}+F_{n+1}^2) \\ (F_nF_{n+2}+F_{n+1}^2) & F_nF_{n+3} & -F_{n+1}L_{n+1} & F_{n-1}F_{n+2} & (F_nF_{n+2}+F_{n+1}^2) \\ 0 & 2F_{n+1}F_{n+2} & 2F_nF_{n+2} & -2F_nF_{n+1} & 0 \\ (F_nF_{n+2}+F_{n+1}^2) & -F_nF_{n+3} & F_{n+1}L_{n+1} & -F_{n-1}F_{n+2} & (F_nF_{n+2}+F_{n+1}^2) \\ -(F_nF_{n+2}+F_{n+1}^2) & F_n^2 & F_{n+1}^2 & F_{n+2}^2 & (F_nF_{n+2}+F_{n+1}^2) \end{vmatrix}$$

is $-(2(F_nF_{n+2}+F_{n+1}^2))^5$.

Solution by Miquel Grau-Sánchez and José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.

Let us denote by Δ the given determinant. Then, from the relation $L_{n+1} = F_n + F_{n+2}$ and setting $a = F_n$, $b = F_{n+1}$, $c = F_{n+2}$, we have $F_{n-1} = b - a$ and

$$\Delta = \begin{vmatrix} ac+b^2 & a^2 & b^2 & c^2 & -(ac+b^2) \\ ac+b^2 & a(b+c) & -b(a+c) & c(b-a) & ac+b^2 \\ 0 & 2bc & 2ac & -2ab & 0 \\ ac+b^2 & -a(b+c) & b(a+c) & -c(b-a) & ac+b^2 \\ -(ac+b^2) & a^2 & b^2 & c^2 & ac+b^2 \end{vmatrix}$$

$$= (ac+b^2)^2 \begin{vmatrix} 1 & a^2 & b^2 & c^2 & -1 \\ 1 & a(b+c) & -b(a+c) & c(b-a) & 1 \\ 0 & 2bc & 2ac & -2ab & 0 \\ 1 & -a(b+c) & b(a+c) & -c(b-a) & 1 \\ -1 & a^2 & b^2 & c^2 & 1 \end{vmatrix}.$$

Making the following row-column transformations $(c_5 + c_1 \longrightarrow c_1)$, $(r_2 + r_4 \longrightarrow r_4)$ and $(-r_1 + r_5 \longrightarrow r_5)$, yields

$$\Delta = (ac+b^2)^2 \begin{vmatrix} 0 & a^2 & b^2 & c^2 & -1 \\ 2 & a(b+c) & -b(a+c) & c(b-a) & 1 \\ 0 & 2bc & 2ac & -2ab & 0 \\ 4 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix},$$

and

$$\Delta = -16(ac+b^2)^2 \begin{vmatrix} a^2 & b^2 & c^2 \\ a(b+c) & -b(a+c) & c(b-a) \\ bc & ac & -ab \end{vmatrix}.$$

Taking into account that $c = a+b$, we have

$$\Lambda := \frac{\Delta}{-16(a^2+b^2+ab)^2} = \begin{vmatrix} a^2 & b^2 & (a+b)^2 \\ a^2+2ab & -b^2-2ab & b^2-a^2 \\ b^2+ab & a^2+ab & -ab \end{vmatrix}.$$

After making the transformation $(r_1 + r_3 \longrightarrow r_3)$, we obtain

$$\Lambda = (a^2+b^2+ab) \begin{vmatrix} a^2 & b^2 & (a+b)^2 \\ a^2+2ab & -b^2-2ab & b^2-a^2 \\ 1 & 1 & 1 \end{vmatrix} = 2(a^2+b^2+ab)^3,$$

from which it immediately follows that

$$\Delta = -16(a^2+b^2+ab)\Lambda = -\Big(2(a^2+b^2+ab)\Big)^5 = -\Big(2(F_nF_{n+2}+F_{n+1}^2)\Big)^5,$$

and the proof is complete.

Also solved by Gökçen Alptekýn and Paul S. Bruckman.

Late acknowledgements:
1. H-621 was also solved by H.-J. Seiffert, who noted that it is essentially the same as H-479.
2. H-627 and H-628 were also solved by Paul S. Bruckman.

Retraction: The proposer of H-638 wishes to retract this problem as it has already appeared as B-1009.

Errata: There are some misprints in the published solution to **H-572**, in Volume **40**, May 2002, page 191. Expression (12) there should be

$$\frac{2\pi}{25}\left[\sin\frac{2\pi}{5} - \sin\frac{8\pi}{5} + \phi\left(\sin\frac{4\pi}{5} - \sin\frac{6\pi}{5}\right)\right].$$