ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
FLORIAN LUCA

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PROBLEMS PROPOSED IN THIS ISSUE

H-739 Proposed by Hideyuki Ohtsuka, Saitama, Japan

Define the generalized Fibonomial coefficient $\binom{n}{k}_{F;m}$ by

$$\binom{n}{k}_{F;m} = \prod_{j=1}^{k} \frac{F_{m(n-j+1)}}{F_{mj}} \quad (\text{for } n \geq k > 0) \quad \text{with} \quad \binom{n}{0}_{F;m} = 1.$$

Prove that

$$\sum_{k=0}^{n} \alpha^{2mn(n-2k)} \binom{n}{k}_{F;2m}^{2} = \sum_{k=0}^{2n} (-1)^{(m+1)(n-k)} \binom{2n}{k}_{F;m}^{2}.$$

H-740 Proposed by Saeid Alikhani, Yazd, Iran and Emeric Deutsch, Brooklyn, New York

Given a simple graph G with vertex set V, a dominating set of G is any subset S of V such that every vertex in $V \backslash S$ is adjacent to at least one vertex in S. Find the number of dominating sets of the path P_n with n vertices.

H-741 Proposed by Charlie Cook, Sumter, South Carolina

If $n \geq 2$ and $m \geq 1$, then

$$m(L_n - F_n)(L_n F_n)^{(m-1)/2} \leq L_n^m - F_n^m,$$

where L_n and F_n are the Lucas and Fibonacci numbers, respectively.

H-742 Proposed by H. Ohtsuka, Saitama, Japan

For positive integers n, m and p with $p < m$ find a closed form expression for

$$\sum_{k_1,\ldots,k_m=1}^{n} F_{2k_1} \cdots F_{2k_m} F_{2(k_1^2+\cdots+k_p^2-k_{p+1}^2-\cdots-k_m^2)}.$$

SOLUTIONS

Convergent Series Involving Fibonacci and Lucas Numbers With Rational Sums

H-713 **Proposed by Hideyuki Ohtsuka, Saitama, Japan**
(Vol. 50, No. 1, February 2012)

Determine

$$(1) \qquad \sum_{k=1}^{\infty} \frac{2^k F_{2^k}}{L_{3\cdot 2^k}} \qquad \text{and} \qquad (2) \qquad \sum_{k=1}^{\infty} \frac{2^k F_{2^k}^3}{L_{2\cdot 2^k} L_{3\cdot 2^k}}.$$

Solution by the proposer

The following identities are known:

(i) $F_{2n} = F_n L_n$;
(ii) $L_n^2 = L_{2n} + 2(-1)^n$;
(iii) $L_{n+m} + (-1)^m L_{n-m} = L_n L_m$;
(iv) $L_{n+m} - (-1)^m L_{n-m} = 5F_n F_m$.

Next, we show the identity

$$\sum_{k=1}^{\infty} \frac{2^k F_{2^k m}}{L_{2^k n} + L_{2^k m}} = \frac{2F_{2m}}{L_{2n} - L_{2m}} \quad (\text{for } n > m \geq 1). \tag{A}$$

Using (i) and (ii), we have

$$\frac{F_{2^k m}}{L_{2^k n} - L_{2^k m}} - \frac{F_{2^k m}}{L_{2^k n} + L_{2^k m}} = \frac{2F_{2^k m} L_{2^k m}}{L_{2^k n}^2 - L_{2^k m}^2} = \frac{2F_{2^{k+1} m}}{L_{2^{k+1} n} - L_{2^{k+1} m}}.$$

Thus,

$$\frac{F_{2^k m}}{L_{2^k n} + L_{2^k m}} = \frac{F_{2^k m}}{L_{2^k n} - L_{2^k m}} - \frac{F_{2^{k+1} m}}{L_{2^{k+1} n} - L_{2^{k+1} m}}.$$

Therefore,

$$\begin{aligned}
\sum_{k=1}^{N} \frac{2^k F_{2^k m}}{L_{2^k n} + L_{2^k m}} &= \sum_{k=1}^{N} \left(\frac{2^k F_{2^k m}}{L_{2^k n} - L_{2^k m}} - \frac{2^{k+1} F_{2^{k+1} m}}{L_{2^{k+1} n} - L_{2^{k+1} m}} \right) \\
&= \frac{2F_{2m}}{L_{2n} - L_{2m}} - \frac{2^{N+1} F_{2^{N+1} m}}{L_{2^{N+1} n} - L_{2^{N+1} m}}.
\end{aligned}$$

Here,

$$\begin{aligned}
\lim_{N\to\infty} \frac{2^{N+1} F_{2^{N+1} m}}{L_{2^{N+1} n} - L_{2^{N+1} m}} &= \lim_{N\to\infty} \frac{2^{N+1}(\alpha^{2^{N+1} m} - \beta^{2^{N+1} m})}{\sqrt{5}(\alpha^{2^{N+1} n} + \beta^{2^{N+1} n} - \alpha^{2^{N+1} m} - \beta^{2^{N+1} m})} \\
&= \lim_{N\to\infty} \frac{2^{N+1} \alpha^{2^{N+1} m}}{\sqrt{5}(\alpha^{2^{N+1} n} - \alpha^{2^{N+1} m})} \\
&= \lim_{N\to\infty} \frac{2^{N+1}}{\sqrt{5}(\alpha^{2^{N+1}(n-m)} - 1)} = 0.
\end{aligned}$$

Therefore, we obtain identity (A).

(1) Using (i), (iii) and (A), we have

$$\sum_{k=1}^{\infty} \frac{2^k F_{2^k}}{L_{3\cdot 2^k}} = \sum_{k=1}^{\infty} \frac{2^k F_{2^k} L_{2^k}}{L_{3\cdot 2^k} L_{2^k}} = \sum_{k=1}^{\infty} \frac{2^k F_{2\cdot 2^k}}{L_{4\cdot 2^k} + L_{2\cdot 2^k}} = \frac{2F_4}{L_8 - L_4} = \frac{3}{20}.$$

(2) Using (i), (iii), (iv) and (A), we have

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{2^k F_{2^k}^3}{L_{2\cdot 2^k} L_{3\cdot 2^k}} &= \sum_{k=1}^{\infty} \frac{2^k F_{2^k}^3 L_{2^k}}{L_{2\cdot 2^k} L_{3\cdot 2^k} L_{2^k}} = \frac{1}{5} \sum_{k=1}^{\infty} \frac{2^k F_{2^k} \cdot (5 F_{2\cdot 2^k} F_{2^k})}{L_{2\cdot 2^k} L_{3\cdot 2^k} L_{2^k}} \\
&= \frac{1}{5} \sum_{k=1}^{\infty} \frac{2^k F_{2^k} (L_{3\cdot 2^k} - L_{2^k})}{L_{2\cdot 2^k} L_{3\cdot 2^k} L_{2^k}} \\
&= \frac{1}{5} \left(\sum_{k=1}^{\infty} \frac{2^k F_{2^k}}{L_{2\cdot 2^k} L_{2^k}} - \frac{2^k F_{2^k}}{L_{3\cdot 2^k} L_{2\cdot 2^k}} \right) \\
&= \frac{1}{5} \sum_{k=1}^{\infty} \left(\frac{2^k F_{2^k}}{L_{3\cdot 2^k} + L_{2^k}} - \frac{2^k F_{2^k}}{L_{5\cdot 2^k} + L_{2^k}} \right) \\
&= \frac{1}{5} \left(\frac{2F_2}{L_6 - L_2} - \frac{2F_2}{L_{10} - L_2} \right) = \frac{7}{300}.
\end{aligned}$$

Also solved by Paul S. Bruckman.

<u>Evaluating a Sum Involving Binomial Coefficients</u>

<u>H-714</u> Proposed by N. Gauthier, The Royal Military College of Canada, Kingston, ON
(Vol. 50, No. 1, February 2012)

Let n be a positive integer. Find a closed–form expression for the following sum:

$$S(n) = \sum_{k=1}^{n} k^2 \binom{2n-2k}{n-k} \binom{2k}{k}.$$

Solution by Helmut Prodinger, Stellenbosch, South Africa

a) Maple can evaluate the sum.

b) A human can proceed like that:
Consider the generating function of the sequence:

$$\begin{aligned}
S &= \sum_{n\geq 0} z^n \sum_{k=0}^{n} k^2 \binom{2n-2k}{n-k} \binom{2k}{k} \\
&= \sum_{k\geq 0} z^k k^2 \binom{2k}{k} \cdot \sum_{k\geq 0} z^k \binom{2k}{k} \\
&= \frac{2z(2z+1)}{(1-4z)^{5/2}} \frac{1}{(1-4z)^{1/2}} = \frac{2z(2z+1)}{(1-4z)^3}.
\end{aligned}$$

Further,

$$\begin{aligned}[z^n]S &= 4[z^{n-2}](1-4z)^{-3} + 2[z^{n-1}](1-4z)^{-3} \\ &= 4^{n-1}\binom{n}{2} + 2\cdot 4^{n-1}\binom{n+1}{2} \\ &= \frac{4^{n-1}}{2}\left(n(n-1)+2(n+1)n\right) = \frac{4^n}{8}n(3n+1).\end{aligned}$$

Also solved by Paul S. Bruckman, Kenneth B. Davenport, E. Killic and I. Akkus (jointly, two solutions), Anastasios Kotronis, Harris Kwong, Ángel Plaza and the proposer. Amos E. Gera provided an equivalent answer without a proof.

<u>Sums of Squares of Tribonacci Numbers</u>

<u>H-715</u> Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 50, No. 1, February 2012)

The Tribonacci numbers T_n satisfy

$$T_0 = 0,\ T_1 = T_2 = 1, \qquad T_{n+3} = T_{n+2} + T_{n+1} + T_n \qquad \text{for} \qquad n \geq 0.$$

Find explicit formulas for

$$(1) \qquad \sum_{k=1}^{n} T_k^2 \qquad \text{and} \qquad (2) \qquad \sum_{k=1}^{n}(T_k^2 - T_{k+1}T_{k-1})^2.$$

Solution by Zbigniew Jakubczyk, Warsaw, Poland

Let

$$S = \sum_{k=1}^{n} T_k^2 \qquad \text{and} \qquad T = \sum_{k=2}^{n} T_k T_{k-2}.$$

We have $T_k = T_{k-1} + T_{k-2} + T_{k-3}$ for $k \geq 3$, so

$$\sum_{k=3}^{n} T_k T_{k-1} = \sum_{k=3}^{n} T_{k-1}^2 + \sum_{k=3}^{n} T_{k-2}T_{k-1} + \sum_{k=3}^{n} T_{k-1}T_{k-3}.$$

We get

$$T_{n-1}T_n - 1 = S - T_n^2 - 1 + A - T_nT_{n-2} \qquad \text{or} \qquad T_n(T_{n-2} + T_{n-1} + T_n) = S + A.$$

Thus,

$$T_nT_{n+1} = S + A. \tag{A}$$

It follows from the recurrence relation that $T_k - T_{k-2} = T_{k-1} + T_{k-3}$. Next,

$$\sum_{k=3}^{n} T_k^2 + \sum_{k=3}^{n} T_{k-2}^2 - 2\sum_{k=3}^{n} T_kT_{k-2} = \sum_{k=3}^{n-1} T_{k-1}^2 + \sum_{k=3}^{n} T_{k-3}^2 + 2\sum_{k=3}^{n} T_{k-1}T_{k-3}.$$

Thus,

$$S - 2 + S - T_{n-1}^2 - T_n^2 - 2A = S - 1 - T_n^2 + S - T_n^2 - T_{n-1}^2 - T_{n-2}^2 + 2(A - T_nT_{n-2}),$$

giving

$$A = \frac{(T_n + T_{n-2})^2 - 1}{4} = \frac{(T_{n+1} - T_{n-1})^2 - 1}{4}. \tag{B}$$

Using (B) in (A), we get

$$S = \frac{4T_nT_{n+1} - (T_{n+1} - T_{n-1})^2 + 1}{4}.$$

To find the second formula, we will consider the sequence

$$R_k = T_k^2 - T_{k-1}T_{k+1} \qquad \text{for} \qquad k \geq 1.$$

We'll prove that

$$R_{k+3} = -R_{k+2} - R_{k+1} + R_k. \tag{C}$$

We have

$$\begin{aligned}
&- R_{k+2} - R_{k+1} + R_k = -T_{k+2}^2 + T_{k+3}T_{k+1} - T_{k+1}^2 + T_{k+2}T_k + T_k^2 - T_{k-1}T_{k+1} \\
&= -T_{k+2}^2 + T_{k+3}T_{k+1} - T_{k+1}^2 + T_{k+2}T_k + T_k^2 - (T_{k+2} - T_{k+1} - T_k)T_{k+1} \\
&= -T_{k+2}^2 + T_{k+3}T_{k+1} + T_{k+2}T_k + T_k^2 - T_{k+2}(T_{k+4} - T_{k+3} - T_{k+2}) + T_{k+1}T_k \\
&= T_{k+3}T_{k+1} + T_{k+2}T_{k+3} - T_{k+2}T_{k+4} + T_k(T_k + T_{k+1} + T_{k+2}) \\
&= T_{k+3}(T_{k+1} + T_{k+2} + T_k) - T_{k+2}T_{k+4} = T_{k+3}^2 - T_{k+2}T_{k+4} = R_{k+3}.
\end{aligned}$$

Let

$$B = \sum_{k=1}^{n} R_k^2 \qquad \text{and} \qquad C = \sum_{k=1}^{n} R_kR_{k+2}.$$

Because $R_k = R_{k+1} + R_{k+2} + R_{k+3}$, we get

$$\sum_{k=1}^{n} R_kR_{k+1} = \sum_{k=1}^{n} R_{k+1}^2 + \sum_{k=1}^{n} R_{k+1}R_{k+2} + \sum_{k=1}^{n} R_{k+1}R_{k+3}.$$

So,

$$R_1R_2 = B + R_{n+1}^2 - R_1^2 + R_{n+1}R_{n+2} + C - R_1R_3 + R_{n+1}R_{n+3}.$$

But $R_1 = 1$, $R_2 = -1$, $R_3 = 0$. Thus,

$$B + C + R_{n+1}(R_{n+1} + R_{n+2} + R_{n+3}) = 0 \qquad \text{giving} \qquad B + C + R_{n+1}R_n = 0. \tag{D}$$

From (C), we have

$$R_{k+3} + R_{k+1} = R_k - R_{k+2},$$

so

$$\sum_{k=1}^{n} R_{k+3}^2 + 2\sum_{k=1}^{n} R_{k+3}R_{k+1} + \sum_{k=1}^{n} R_{k+1}^2 = \sum_{k=1}^{n} R_k^2 - 2\sum_{k=1}^{n} R_kR_{k+2} + \sum_{k=1}^{n} R_{k+2}^2,$$

or

$$R_{n+3}^2 + 2(C + R_{n+1}R_{n+3} - R_2R_3) + R_{n+1}^2 = R_1^2 - 2C + R_3^2,$$

which gives

$$C = \frac{1 - (R_{n+3} + R_{n+1})^2}{4}.$$

Using this last equation and (D), we get

$$B = \frac{(R_{n+3} + R_{n+1})^2 - 1 - 4R_nR_{n+1}}{4}.$$

Also solved by Paul S. Bruckman, Kenneth B. Davenport, Nazmiye Yilmaz, Yasin Yazlik and Necati Taskara (the last three jointly), and the proposer. Abbas Rouholamini provided an equivalent answer for (1) without a proof.

Convolutions with Catalan Numbers

H-716 Proposed by N. Gauthier, The Royal Military College of Canada, Kingston, ON
(Vol. 50, No. 2, May 2012)

Let n be a nonnegative integer and let C_n be the nth Catalan number. Prove the following identity:

$$\sum_{k=0}^{n} k^3 C_{n-k} C_k = \frac{n}{2}((n^2+3n+3)C_{n+1} - 3 \cdot 4^n).$$

Solution by Hideyuki Ohtsuka, Saitama, Japan

The following identity is well-known

$$\sum_{k=0}^{n} C_k C_{n-k} = C_{n+1}. \tag{1}$$

We have

$$\sum_{k=0}^{n} k C_{n-k} C_k = \sum_{k=0}^{n} (n-k) C_k C_{n-k}.$$

Therefore,

$$\sum_{k=0}^{n} k C_{n-k} C_k = \frac{n}{2} \sum_{k=0}^{n} C_{n-k} C_k = \frac{n}{2} C_{n+1}, \tag{2}$$

(by (1)). We have

$$\begin{aligned} 4^n &= \sum_{k=0}^{n} \binom{2n-2k}{n-k} \binom{2k}{k} \qquad \text{(see [1](5.39))} \\ &= \sum_{k=0}^{n} (n-k+1)(k+1) C_{n-k} C_k \\ &= (n+1) \sum_{k=0}^{n} C_{n-k} C_k + n \sum_{k=0}^{n} k C_{n-k} C_k - \sum_{k=0}^{n} k^2 C_{n-k} C_k \\ &= \frac{n^2+2n+2}{2} C_{n+1} - \sum_{k=0}^{n} k^2 C_{n-k} C_k \qquad \text{(by (1) and (2))}. \end{aligned}$$

Thus,

$$\sum_{k=0}^{n} k^2 C_{n-1} C_k = \frac{n^2+2n+2}{2} C_{n+1} - 4^n. \tag{3}$$

We have

$$\sum_{k=0}^{n} k^3 C_{n-k} C_k = \sum_{k=0}^{n} (n-k)^3 C_k C_{n-k}.$$

Therefore, we have

$$\begin{aligned}\sum_{k=0}^{n} k^3 C_{n-k} C_k &= \frac{1}{2}\left(n^3 \sum_{k=0}^{n} C_{n-k} C_k - 3n^2 \sum_{k=0}^{n} k C_{n-k} C_k + 3n \sum_{k=0}^{n} k^2 C_{n-k} C_k\right) \\ &= \frac{n}{2}\left((n^2+3n+3)C_{n+1} - 3\cdot 4^n\right),\end{aligned}$$

by (1), (2) and (3).

References

[1] R. L. Graham, D. E. Knuth and O. Patashnik, *Concrete Mathematics*, Addison-Wesley, 1989.

Also solved by Wouter Cames van Batenburg, Paul S. Bruckman, M. N. Deshpande and the proposer.

Obituary. The Editor is deeply saddened to announce that the long time contributor and friend of this section, Paul S. Bruckman, passed away on May 3, 2013. This Department will miss Paul's contributions some of which are described in the preamble of the Advanced Problem Section of FQ volume **49.3**, (August, 2011) which itself is a tribute to Paul.
