ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
RUSS EULER AND JAWAD SADEK

Please submit all new problem proposals and their solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468, or by email at reuler@nwmissouri.edu. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by August 15, 2011. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2,\ \beta = (1-\sqrt{5})/2,\ F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1081 Proposed by Br. J. M. Mahon, Kensington, Australia.

Prove that

$$F_{n+1}^2 F_{n-1}^2 - F_n^4 = \frac{(-1)^n}{5}\left[2L_{2n} + (-1)^n\right].$$

B-1082 **Proposed by Ángel Plaza and Sergio Falcón, Universidad de Las Palmas de Gran Canaria, Spain.**

The k-Fibonacci numbers $F_n = F_{k,n}$ satisfy $F_{k,n+2} = kF_{k,n+1} + F_{k,n}$, $F_{k,0} = 0$, $F_{k,1} = 1$. Let $m > 1$ be a fixed integer. Prove that

$$\sum_{r=1}^{\infty} \frac{(-1)^{r-1} F_{2mr+m}}{F_{mr}^2 F_{mr+m}^2} = \frac{1}{F_m^3}.$$

B-1083 **Proposed by Br. J. M. Mahon, Kensington, Australia.**

Find a closed form for

$$\sum_{j=0}^{[\frac{n}{2}]} (-1)^j \frac{n}{n-j} \binom{n-j}{j} F_{2n-3j}.$$

B-1084 **Proposed by José Luis Díaz-Barrero, Barcelona, Spain.**

Let n be a positive integer. Prove that

$$\left(\sum_{k=1}^{n} \frac{F_k^2}{\sqrt{1+F_k^2}} \right) \left(\prod_{k=1}^{n} (1+F_k^2) \right)^{1/2n} \leq F_n F_{n+1}.$$

B-1085 **Proposed by Carsten Elsner and Martin Stein, University of Applied Sciences, Hannover, Germany.**

Let $(a_n)_{n\geq 1}$ be a sequence of positive integers. Let $q_0 = 1$, $q_1 = a_1$ and $q_n = a_n q_{n-1} + q_{n-2}$ for $n \geq 2$. Prove that

$$\frac{q_0 + q_1 + \cdots + q_{m-1}}{q_m} \leq \frac{F_{m+2} - 1}{F_{m+1}}$$

for $m \geq 1$.

SOLUTIONS

Sum of Products

B-1061 Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 48.1, February 2010)

Show that, for all positive integers n,

$$\sum_{k=1}^{n}(-1)^{\lfloor\frac{k}{2}\rfloor}\frac{F_k}{F_{k+1}}\left(\prod_{j=k}^{n}F_j\right)^2=(-1)^{\lfloor\frac{n-1}{2}\rfloor}\frac{F_n}{F_{n+1}}$$

where $\lfloor\cdot\rfloor$ denotes the greatest integer function.

Solution by Jay Hendel, Towson University, Towson, MD and Jaroslav Seibert, University of Pardubice, The Czech Republic (independently).

We will prove the given formula by induction on n. Let $S(n)$ denote the sum on the left side of the equality. It is easy to see that $S(1)=\frac{F_1}{F_2}F_1^2=1$. Suppose that the equality is true for a positive integer n, and show that it is also true for $n+1$.

$$\begin{aligned}S(n+1)&=\left(S(n)+(-1)^{\lfloor\frac{n+1}{2}\rfloor}\frac{F_{n+1}}{F_{n+2}}\right)F_{n+1}^2=\left((-1)^{\lfloor\frac{n-1}{2}\rfloor}\frac{F_n}{F_{n+1}}+(-1)^{\lfloor\frac{n+1}{2}\rfloor}\frac{F_{n+1}}{F_{n+2}}\right)F_{n+1}^2\\&=(-1)^{\lfloor\frac{n-1}{2}\rfloor}\frac{F_{n+1}}{F_{n+2}}\left(\frac{F_nF_{n+2}}{F_{n+1}^2}-1\right)F_{n+1}^2=(-1)^{\lfloor\frac{n-1}{2}\rfloor}\frac{F_{n+1}}{F_{n+2}}(F_nF_{n+2}-F_{n+1}^2)\\&=(-1)^{\lfloor\frac{n-1}{2}\rfloor}(-1)^{n+1}\frac{F_{n+1}}{F_{n+2}}\end{aligned}$$

using Cassini's identity ([1]; identity (29)).

If n is even then $(-1)^{\lfloor\frac{n-1}{2}\rfloor}(-1)^{n+1}=(-1)^{\lfloor\frac{n+1}{2}\rfloor}=(-1)^{\lfloor\frac{n}{2}\rfloor}$ and if n is odd then $(-1)^{\lfloor\frac{n-1}{2}\rfloor}(-1)^{n+1}=(-1)^{\lfloor\frac{n-1}{2}\rfloor}=(-1)^{\lfloor\frac{n}{2}\rfloor}$, which completes the proof.

References

[1] S. Vajda, *Fibonacci and Lucas Numbers, and the Golden Section*, Chichester, Ellis Horwood Ltd., 1989.

Also solved by Paul S. Bruckman, Sergio Falcón and Ángel Plaza (jointly), G. C. Greubel, and the proposer.

Falcón and Plaza have a generalization of this inequality that will appear as a separate proposal.

A Lot of Sums!

B-1062 Proposed by M. N. Deshpande, Nagpur, India (Vol. 48.1, February 2010)

Let $g(n)=F_n^2+F_{n+1}^2+F_{n+2}^2$ for $n\geq 0$. For every $n\geq 2$, show that $[4g(n+2)-7g(n+1)-9g(n)]/4$ is a product of two consecutive Fibonacci numbers.

Solution by Rebecca A. Hillman, University of South Carolina Sumter, Sumter, SC 29150

Substituting $g(n)$ into an expression and using the basic formula, $F_{n+2} = F_{n+1} + F_n$ for various values of n, it is seen that

$$\begin{aligned}\frac{1}{4}[4g(n+2) - 7g(n+1) - 9g(n)] &= \frac{1}{4}[4F_{n+4}^2 - 3F_{n+3}^2 - 12F_{n+2}^2 - 16F_{n+1}^2 - 9F_n^2]\\ &= \frac{1}{4}[4(3F_{n+1} + 2F_n)^2 - 3(2F_{n+1} + F_n)^2 - 12(F_{n+1} + F_n)^2\\ &\quad - 16F_{n+1}^2 - 9F_n^2]\\ &= \frac{1}{4}[-4F_{n+1}^2 + 12F_{n+1}F_n - 8F_n^2]\\ &= -[F_{n+1}^2 - 3F_{n+1}F_n + 2F_n^2]\\ &= -(F_{n+1} - 2F_n)(F_{n+1} - F_n)\\ &= F_{n-2}F_{n-1}.\end{aligned}$$

Therefore, $[4g(n+2)-7g(n+1)-9g(n)]/4$ is the product of two consecutive Fibonacci numbers, namely, $F_{n-2}Fn - 1$.

Also solved by Brian Beasley, Scott Brown, Paul S. Bruckman, Charles Cook, Kenneth Davenport, Sergio Falcón and Ángel Plaza (jointly), G. C. Greubel, Jay Hendel, Harris Kwong, Jaroslav Seibert, David Terr, and the proposer.

Another Sum and a Product

B-1063 Proposed by José Luis Díaz-Barrero, Universidad Politécnica de Cataluña, Barcelona, Spain (Vol. 48.1, February 2010)

Let n be a positive integer. Prove that

$$1 + 8\sum_{k=1}^{n} \frac{F_{2k}^2}{F_k^2 + L_k^2} < \frac{4}{3}\ (F_nF_{n+1} + 1)(L_nL_{n+2} - 1).$$

Solution by Ángel Plaza and Sergio Falcón (jointly), Department of Mathematics, Universidad de Las Palmas de Gran Canaria, Spain

For $n = 1$,

$$\begin{aligned}1 + 8\frac{F_2^2}{F_1^2 + L_1^2} &< \frac{4}{3}(F_1F_2 + 1)(L_1L_3 - 1)\\ 1 + 8\frac{1}{1+1} &< \frac{4}{3}\cdot 6\\ 1 + 4 &< 8.\end{aligned}$$

In the following we use that $F_{2k} = F_k L_k$:

$$1 + 8\sum_{k=1}^{n} \frac{F_2^2}{F_k^2 + L_k^2} = 1 + 8\sum_{k=1}^{n} \frac{F_k^2 L_k^2}{F_k^2 + L_k^2} < 1 + 8\sum_{k=1}^{n} \frac{F_k^2 L_k^2}{2F_k^2}$$

$$< 1 + 4\sum_{k=1}^{n} L_k^2 = 1 + 4(L_n L_{n+1} - 2)$$

$$= 1 + 4(L_n L_{n+2} - L_n^2 - 2) < 4(L_n L_{n+2} - 1).$$

Taking into account that for $n > 1$, $4 \leq \frac{4}{3}(F_n F_{n+1} + 1)$, the proof is done.

It should be noted that for $n \geq 2$, the following stronger inequality holds.

For any integer $n \geq 2$,

$$1 + 8\sum_{k=1}^{n} \frac{F_{2k}^2}{F_k^2 + L_k^2} < L_n L_{n+2}.$$

Also solved by Paul S. Bruckman, Jay Hendel, and the proposer.

Generalized Fibonacci Polynomials ... Again!

B-1064 Proposed by N. Gauthier, Kingston, ON, Canada (Vol. 48.1, February 2010)

For $a \neq 0$, let $f_0 = 0$, $f_1 = 1$, and $f_{n+2} = af_{n+1} + f_n$ for $n \geq 0$. If n is a positive integer, find a closed-form expression for

$$\sum_{k=0}^{n-1} f_k^3.$$

Solution by Harris Kwong, SUNY Fredonia, Fredonia, NY

Let $\alpha = (a + \sqrt{a^2+4})/2$, and $\beta = (a - \sqrt{a^2+4})/2$. The Binet's form for f_k is

$$f_k = \frac{\alpha^k - \beta^k}{\alpha - \beta}.$$

Notice that $\alpha + \beta = a$, $\alpha\beta = -1$, and $\alpha - \beta = \sqrt{a^2+4}$. Hence,

$$\sum_{k=0}^{n-1} f_k^3 = \sum_{k=0}^{n-1} \left(\frac{\alpha^k - \beta^k}{\alpha - \beta}\right)^3 = \frac{1}{a^2+4}\sum_{k=0}^{n-1} \frac{(\alpha^k - \beta^k)^3}{\alpha - \beta}.$$

From

$$(\alpha^k - \beta^k)^3 = \alpha^{3k} - 3(\alpha\beta)^k(\alpha^k - \beta^k) - \beta^{3k} = \alpha^{3k} - \beta^{3k} + 3[(-\beta)^k - (-\alpha)^k],$$

we obtain

$$\sum_{k=0}^{n-1} (\alpha^k - \beta^k)^3 = \frac{1 - \alpha^{3n}}{1 - \alpha^3} - \frac{1 - \beta^{3n}}{1 - \beta^3} + 3\left[\frac{1 - (-\beta)^n}{1 + \beta} - \frac{1 - (-\alpha)^n}{1 + \alpha}\right].$$

Since $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)=a^3+3a$, we find

$$\begin{aligned}\frac{1-\alpha^{3n}}{1-\alpha^3}-\frac{1-\beta^{3n}}{1-\beta^3}&=\frac{(1-\beta^3)(1-\alpha^{3n})-(1-\alpha^3)(1-\beta^{3n})}{(1-\alpha^3)(1-\beta^3)}\\&=\frac{\alpha^3-\beta^3-(\alpha^{3n}-\beta^{3n})+(\alpha\beta)^3(\alpha^{3n-3}-\beta^{3n-3})}{1-(\alpha^3+\beta^3)+(\alpha\beta)^3}\\&=\frac{(\alpha-\beta)(f_{3n}+f_{3n-3}-f_3)}{a(a^2+3)},\end{aligned}$$

and

$$\begin{aligned}\frac{1-(-\beta)^n}{1+\beta^3}-\frac{1-(-\alpha)^n}{1+\alpha}&=\frac{(1+\alpha)[1-(-\beta)^n]-(1+\beta)[(1-(-\alpha)^n)]}{(1+\alpha)(1+\beta)}\\&=\frac{\alpha-\beta+(-1)^n[(\alpha^n-\beta^n)-(\alpha^{n-1}-\beta^{n-1})]}{1+(\alpha+\beta)+\alpha\beta}\\&=\frac{(\alpha-\beta)[f_1+(-1)^n(f_n-f_{n-1})]}{a}.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{k=0}^{n-1}f_k^3&=\frac{1}{a^2+4}\left(\frac{f_{3n}+f_{3n-3}-f_3}{a(a^2+3)}+\frac{3[f_1+(-1)^n(f_n-f_{n-1})]}{a}\right)\\&=\frac{f_{3n}+f_{3n-3}+3(-1)^n(a^2+3)(f_n-f_{n-1})+2a^2+8}{a(a^2+3)(a^2+4)}.\end{aligned}$$

Also solved by Paul S. Bruckman, Charles Cook, Sergio Falcón and Ángel Plaza (jointly), G. C. Greubel, Jay Hendel, and the proposer.

A Sum of Pell Numbers

B-1065 Proposed by Br. J. Mahon, Australia (Vol. 48.1, February 2010)

The Pell numbers P_n satisfy $P_{n+2}=2P_{n+1}+P_n$, $P_0=0$, $P_1=1$. Prove that

$$\sum_{r=1}^{\infty}\frac{(-1)^{r-1}P_{6r+3}}{P_{3r}^2P_{3r+3}^2}=\frac{1}{125}.$$

Solution by Paul S. Bruckman, Nanaimo, BC, Canada

Given N natural, let $T(N)=\sum_{r=1}^{N}\frac{(-1)^{r-1}P_{6r+3}}{P_{3r}^2P_{3r+3}^2}$, and let $T=\sum_{r=1}^{\infty}\frac{(-1)^{r-1}P_{6r+3}}{P_{3r}^2P_{3r+3}^2}=\lim T(N)$ as $N\to\infty$, provided such limit exists. The Pell numbers satisfy the following relations for $n=0,1,2,\ldots$,

$$P_n=\frac{u^n-v^n}{u-v},\quad\text{where}\quad u=1+\sqrt{2},v=1-\sqrt{2};\tag{1}$$

$$P_n^2+P_{n+1}^2=P_{2n+1};\tag{2}$$

$$P_{3n}^2+P_{3n+3}^2=5P_{6n+3};\tag{3}$$

$$\frac{1}{P_n^2} \sim \frac{8}{u^{2n}} \to 0 \quad \text{as} \quad n \to \infty. \tag{4}$$

Equations (1), (2), and (4) are well-known; (3) is derived similarly to (2) (by expanding the Binet formulas) and using the fact that $P_3 = 5$.

Then

$$T(N) = \sum_{r=1}^{N} \frac{(-1)^{r-1}\{P_{3r}^2 + P_{3r+3}^2\}}{5P_{3r}^2 P_{3r+3}^2} = \frac{1}{5}\sum_{r=1}^{N}\left\{\frac{(-1)^{r-1}}{P_{3r}^2} - \frac{(-1)^r}{P_{3r+3}^2}\right\},$$

a telescoping sum that is readily evaluated as

$$T(N) = \frac{1}{5P_3^2} - \frac{(-1)^N}{5P_{3N+3}^2} = \frac{1}{125} - \frac{(-1)^N}{5P_{3N+3}^2}. \tag{5}$$

Using (4), we see that T does exist, with $T = \frac{1}{125}$.

Also solved by Sergio Falcón and Ángel Plaza (jointly), Kenneth Davenport, Jaroslav Seibert, and the proposer.

A late solution to Problem B-1058 by Zbigniew Jakubczyk was received.