

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
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Please submit all new problem proposals and their solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468, or by email at reuler@nwmissouri.edu. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by August 15, 2013. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

The content of the problem sections of *The Fibonacci Quarterly* are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.$$

Also, $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1114 (Correction) Proposed by D. M. Băţineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania

If $f : R^+ \rightarrow R^+$ such that $f(x) > x$ for all $x \in R^+$, prove that

$$\sum_{k=1}^n f((1 + F_k^2)^2) > 4F_n F_{n+1}$$

for all positive integers n .

B-1121 Proposed by D. M. Bătinețu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade General School, Buzău, Romania

Prove that

$$n + 4 + 4F_n F_{n+1} > 4F_{n+2} \quad (1)$$

and

$$n + 4 + 4L_n L_{n+1} > 4L_{n+2} \quad (2)$$

for any positive integer n .

B-1122 Proposed by Harris Kwong, SUNY Fredonia, Fredonia, NY

Prove that, given any integer $r \geq 4$, if $\gcd(2r - 1, r^2 - r - 1) = 1$, then

$$F_{n+\phi(r^2-r-1)} \equiv F_n \pmod{r^2 - r - 1}$$

for all nonnegative integers n . Here, ϕ denotes Euler's phi-function.

B-1123 Proposed by José Luis Díaz-Barrero, BARCELONA TECH, Barcelona, Spain and Mihály Bencze, Braşov, Romania

Let $n \geq 2$ be a positive integer. Prove that

$$\frac{1}{n} \sum_{k=1}^n \frac{F_k^3}{F_n F_{n+1} - F_k^2} \geq \frac{1}{n-1} \sqrt{\frac{1}{F_{n+2} - 1} \sum_{k=1}^n F_k^3}.$$

B-1124 Proposed by D. M. Bătinețu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade General School, Buzău, Romania

Prove that

$$\sum_{k=1}^n \left(\frac{F_k}{F_{k+3}} + \frac{F_{k+1}}{2F_k + F_{k+1}} \right) > \frac{n}{2} \quad (1)$$

$$\sum_{k=1}^n \left(\frac{L_k}{L_{k+3}} + \frac{L_{k+1}}{2L_k + L_{k+1}} \right) > \frac{n}{2} \quad (2)$$

for any positive integer n .

B-1125 Proposed by D. M. Bătinețu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade General School, Buzău, Romania

Prove that

$$\frac{(L_1^2 + 1)(L_2^2 + 1)}{L_1 L_2 + 1} + \frac{(L_2^2 + 1)(L_3^2 + 1)}{L_2 L_3 + 1} + \dots + \frac{(L_{n-1}^2 + 1)(L_n^2 + 1)}{L_{n-1} L_n + 1} + \frac{(L_n^2 + 1)(L_1^2 + 1)}{L_n L_1 + 1} \geq 2L_{n+2} - 6,$$

for any positive integer n .

The Ubiquitous AM-GM Inequality

B-1101 Proposed by D. M. Bătinețu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania
(Vol. 50.1, February 2012)

Prove that

$$\arctan \sqrt{\frac{F_n^2 + F_{n+1}^2}{2}} + \arctan \sqrt{\frac{L_n^2 + L_{n+1}^2}{2}} \geq \arctan \frac{F_{n+2}}{2} + \arctan \frac{L_{n+2}}{2}.$$

Almost all solvers provided a version of the following solution.

$$\begin{aligned} (F_n - F_{n+1})^2 \geq 0 &\Rightarrow F_n^2 + F_{n+1}^2 \geq 2F_n F_{n+1} \Rightarrow 2(F_n^2 + F_{n+1}^2) \geq (F_n + F_{n+1})^2 = F_{n+2}^2 \\ &\Rightarrow \frac{(F_n^2 + F_{n+1}^2)}{2} \geq \frac{F_{n+2}^2}{4} \Rightarrow \sqrt{\frac{F_n^2 + F_{n+1}^2}{2}} \geq \frac{F_{n+2}}{2} \end{aligned}$$

and since the arctan function is increasing, it follows that

$$\arctan \sqrt{\frac{F_n^2 + F_{n+1}^2}{2}} \geq \arctan \frac{F_{n+2}}{2}.$$

An identical argument works for the Lucas inequality. Thus, adding the two inequalities yields the desired result.

Also solved by Paul S. Bruckman, Charles K. Cook, Kenneth B. Davenport, Robinson Higueta and Alexander Ramirez (jointly), Zbigniew Jakubczyk (student), Harris Kwong, ONU-Solve Problem Group, Ángel Plaza, Jaroslav Seibert, and the proposer.

From the Weighted AM-GM Inequality

B-1102 Proposed by Diana Alexandrescu, University of Bucharest, Bucharest, Romania and José Luis Díaz-Barrero, BARCELONA TECH, Barcelona, Spain
(Vol. 50.1, February 2012)

Let n be a positive integer. Prove that

$$\left(\frac{\sqrt[3]{F_n^2} + \sqrt[3]{F_{n+1}^2}}{\sqrt[3]{F_{n+2}^2}} \right) \left(\frac{\sqrt[3]{L_n^2} + \sqrt[3]{L_{n+1}^2}}{\sqrt[3]{L_{n+2}^2}} \right) < \sqrt[3]{4}.$$

Solution by Zbigniew Jakubczyk (student), Warsaw, Poland.

Since the function $f(x) = \sqrt[3]{x^2}$ is concave for all $x \geq 0$, Jensen's inequality yields

$$\sqrt[3]{\frac{(x+y)^2}{4}} > \frac{\sqrt[3]{x^2} + \sqrt[3]{y^2}}{2},$$

for $x \geq 0, y \geq 0, x \neq y$.

Under the same conditions, this inequality can be equivalently written as

$$\sqrt[3]{x^2} + \sqrt[3]{y^2} < \sqrt[3]{2(x+y)^2}.$$

For $x = F_n$ and $y = F_{n+1}$, we obtain

$$\sqrt[3]{F_n^2} + \sqrt[3]{F_{n+1}^2} < \sqrt[3]{2(F_n + F_{n+1})^2} = \sqrt[3]{2F_{n+2}^2}.$$

This implies

$$\frac{\sqrt[3]{F_n^2} + \sqrt[3]{F_{n+1}^2}}{\sqrt[3]{F_{n+2}^2}} < \sqrt[3]{2}. \tag{1}$$

Similarly, letting $x = L_n$ and $y = L_{n+1}$ yields

$$\frac{\sqrt[3]{L_n^2} + \sqrt[3]{L_{n+1}^2}}{\sqrt[3]{L_{n+2}^2}} < \sqrt[3]{2}. \tag{2}$$

Multiplying inequalities (1) and (2) yields the desired result.

Also solved by Paul S. Bruckman, Charles K. Cook, Kenneth B. Davenport, Robinson Higuera (student), ONU-Solve Problem Group, Ángel Plaza, Jaroslav Seibert, David Stone and John Hawkins (jointly), and the proposer.

An Extension to Negative Subscripts

B-1103 Proposed by Hideyuki Ohtsuka, Saitama, Japan
(Vol. 50.1, February 2012)

If $a + b + c = 0$ and $abc \neq 0$, find the value of

$$\frac{L_a L_b L_c}{F_a F_b F_c} \left(\frac{F_a}{L_a} + \frac{F_b}{L_b} + \frac{F_c}{L_c} \right).$$

Solution by ONU-Solve Problem Group, Ohio Northern University

We will show that if $a + b + c = 0$ and $abc \neq 0$, then

$$\frac{L_a L_b L_c}{F_a F_b F_c} \left(\frac{F_a}{L_a} + \frac{F_b}{L_b} + \frac{F_c}{L_c} \right) = -5. \tag{1}$$

After a straightforward algebraic manipulation, (1) is equivalent to

$$L_a L_b F_c + L_a F_b L_c + F_a L_b L_c + 5F_a F_b F_c = 0. \tag{2}$$

By using $c = -(a + b)$ together with the formulas for the natural extensions of the Fibonacci and Lucas numbers to negative subscripts $F_{-n} = (-1)^{n-1}F_n$ and $L_{-n} = (-1)^n L_n$ we can rewrite (2) in the following form

$$L_a F_b L_{a+b} + F_a L_b L_{a+b} = L_a L_b F_{a+b} + 5F_a F_b F_{a+b}. \tag{3}$$

By using the explicit formulas $F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$ and $L_n = \alpha^n + \beta^n$, where $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$, we have

$$L_a F_b + F_a L_b = 2F_{a+b}, \tag{4}$$

and

$$L_a L_b + 5F_a F_b = 2L_{a+b} \tag{5}$$

Indeed,

$$L_a F_b + F_a L_b = \frac{1}{\sqrt{5}}(\alpha^a + \beta^a)(\alpha^b - \beta^b) + \frac{1}{\sqrt{5}}(\alpha^a - \beta^a)(\alpha^b + \beta^b) = \frac{2}{\sqrt{5}}(\alpha^{a+b} - \beta^{a+b}) = 2F_{a+b}$$

and

$$L_a L_b + 5F_a F_b = (\alpha^a + \beta^a)(\alpha^b + \beta^b) + (\alpha^a - \beta^a)(\alpha^b - \beta^b) = 2(\alpha^{a+b} + \beta^{a+b}) = 2L_{a+b}.$$

We now use the identities (4) and (5) in order to prove (3). Indeed, from (4), the left-hand side of (3) becomes $(L_a F_b + F_a L_b)L_{a+b} = 2F_{a+b}L_{a+b}$. On the other hand, from (5), the right-hand side of (3) becomes $(L_a L_b + 5F_a F_b)F_{a+b} = 2L_{a+b}F_{a+b}$. This concludes the proof of the desired identity.

Also solved by **Paul S. Bruckman, Charles K. Cook, Robinson Higueta (student), Zbigniew Jakubczyk (student), Fabian Maple, Ángel Plaza, Jaroslav Seibert, and the proposer.**

A Symmetrical Identity

B-1104 Proposed by **Javier Sebastián Cortés (student), Universidad Distrital Francisco José de Caldas, Bogotá, Colombia**
(Vol. 50.1, February 2012)

Prove that

$$F_{n+2k(k+1)} \sum_{i=0}^{2k} L_{n+2(k+1)i} = L_{n+2k(k+1)} \sum_{i=0}^{2k} F_{n+2(k+1)i}.$$

Solution by Robinson Higueta (student), Universidad de Antioquia, Colombia.

We claim that

$$\sum_{i=0}^{2k} F_{2(k+1)(k-i)} = 0. \tag{1.1}$$

Indeed,

$$\begin{aligned} \sum_{i=0}^{2k} F_{2(k+1)(k-i)} &= \sum_{i=0}^k F_{2(k+1)(k-i)} + \sum_{i=k}^{2k} F_{2(k+1)(k-i)} = \sum_{i=0}^k F_{2(k+1)i} + \sum_{i=0}^k F_{2(k+1)(-i)} \\ &= \sum_{i=0}^k F_{2(k+1)i} + \sum_{i=0}^k (-1)^{2(k+1)i+1} F_{2(k+1)i} = \sum_{i=0}^k F_{2(k+1)i} - \sum_{i=0}^k F_{2(k+1)i} = 0. \end{aligned}$$

From [1, page 92] we know that $F_m L_n = F_{n+m} + (-1)^n F_{m-n}$. This implies that

$$F_{n+2k(k+1)} L_{n+2i(k+1)} = F_{2n+2(k+1)(i+k)} + (-1)^n F_{2(k+1)(k-i)}, \tag{1.2}$$

and

$$\begin{aligned} L_{n+2k(k+1)}F_{n+2i(k+1)} &= F_{2n+2(k+1)(i+k)} + (-1)^n F_{-2(k+1)(k-i)}, \\ &= F_{2n+2(k+1)(i+k)} + (-1)^{n+1} F_{2(k+1)(k-i)}. \end{aligned} \tag{1.3}$$

Thus, from (1.1), (1.2), and (1.3) we obtain

$$F_{n+2k(k+1)} \sum_{i=0}^{2k} L_{n+2(k+1)i} = \sum_{i=0}^{2k} F_{2n+2(k+1)(i+k)} + (-1)^n \sum_{i=0}^{2k} F_{2(k+1)(k-i)}.$$

This and (1.1) imply that

$$\begin{aligned} F_{n+2k(k+1)} \sum_{i=0}^{2k} L_{n+2(k+1)i} &= \sum_{i=0}^{2k} F_{2n+2(k+1)(i+k)} + (-1)^{n+1} \sum_{i=0}^{2k} F_{2(k+1)(k-i)} \\ &= L_{n+2k(k+1)} \sum_{i=0}^{2k} F_{n+2(k+1)i}. \end{aligned}$$

REFERENCES

[1] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley, New York, 2001.

Also solved by Paul S. Bruckman, Kenneth B. Davenport, Harris Kwong, Ángel Plaza, Jaroslav Seibert, and the proposer.

Fibonomial Coefficients

B-1105 Proposed by Paul S. Bruckman, Nanaimo, BC, Canada
(Vol. 50.1, February 2012)

Let

$$G_m(x) = \sum_{k=0}^{m+1} (-1)^{k(k+1)/2} \begin{bmatrix} m+1 \\ k \end{bmatrix}_F x^{m+1-k}, m = 0, 1, 2, \dots,$$

where $\begin{bmatrix} m+1 \\ k \end{bmatrix}_F$ is the *Fibonomial coefficient* $\frac{F_1 F_2 F_3 \dots F_{m+1}}{(F_1 F_2 F_3 \dots F_k)(F_1 F_2 F_3 \dots F_{m+1-k})}$, $1 \leq k \leq m$; also define

$$\begin{bmatrix} m+1 \\ 0 \end{bmatrix}_F = \begin{bmatrix} m+1 \\ m+1 \end{bmatrix}_F = 1.$$

Let $G_m(1) = U_m$. Prove the following, for $n = 0, 1, 2, \dots$:

- (a) $U_{4n} = 0$;
- (b) $U_{4n+2} = 2(-1)^{n+1} \{L_1 L_2 L_3 \dots L_{2n+1}\}^2$;
- (c) $U_{2n+1} = (-1)^{(n+1)(n+2)/2} \{L_1 L_3 L_5 \dots L_{2n+1}\}$.

Solution by E. Kilic and I. Akkus (jointly).

From [1], we have

$$G_m(x) = \sum_{k=0}^{m+1} (-1)^{k(k+1)/2} \begin{bmatrix} m+1 \\ k \end{bmatrix}_F x^{m+1-k} = \prod_{j=0}^m (1 - \alpha^j \beta^{m-j}),$$

THE FIBONACCI QUARTERLY

where $\begin{bmatrix} n \\ k \end{bmatrix}_F$ stands for the usual Fibonomial coefficients, $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$, $\alpha + \beta = 1$, and $\alpha\beta = -1$.

(a) Notice that $G_{4n}(1) = U_{4n}$. Since

$$U_{4n} = \sum_{k=0}^{4n+1} (-1)^{k(k+1)/2} \begin{bmatrix} 4n+1 \\ k \end{bmatrix}_F = \prod_{j=0}^{4n} (1 - \alpha^j \beta^{4n-j})$$

includes the factor $(1 - (\alpha\beta)^{2n}) = 0$, $U_{4n} = 0$.

(b) Using the Binet formula for the Lucas numbers, we obtain

$$\begin{aligned} U_{4n+2} &= \sum_{k=0}^{4n+3} (-1)^{k(k+1)/2} \begin{bmatrix} 4n+3 \\ k \end{bmatrix}_F = \prod_{i=0}^{4n+2} (1 - \alpha^i \beta^{4n+2-i}) \\ &= 2 \prod_{i=1}^{2n+1} (1 + (-1)^i \alpha^{2i})(1 + (-1)^i \beta^{2i}) \\ &= 2 \prod_{i=1}^{2n+1} (-1)^i (\alpha^{2i} + 2(-1)^i + \beta^{2i}) \\ &= 2(-1)^{n+1} \prod_{i=1}^{2n+1} (\alpha^i + \beta^i)^2 \\ &= 2(-1)^{n+1} \prod_{i=1}^{2n+1} L_i^2. \end{aligned}$$

(c) For $m = 2n + 1$,

$$\begin{aligned} U_{2n+1} &= \sum_{k=0}^{2n+2} (-1)^{k(k+1)/2} \begin{bmatrix} 2n+2 \\ k \end{bmatrix}_F = \prod_{j=0}^{2n+1} (1 - \alpha^j \beta^{2n+1-j}) \\ &= \prod_{j=0}^n (-1)^{j+1} (\alpha^{2n+1-2j} + \beta^{2n+1-2j}) \\ &= (-1)^{\binom{n+2}{2}} \prod_{j=0}^n L_{2j+1}. \end{aligned}$$

REFERENCES

[1] L. Carlitz, *The characteristic polynomial of a certain matrix of binomial coefficients*, The Fibonacci Quarterly, **3.2** (1965), 81–89.

All solvers gave, more or less, a similar proof.

Also solved by Harris Kwong, Ángel Plaza and Sergio Falcón (jointly), and the proposer.

We wish to belatedly acknowledge the solution to problem B-1099 by Amos Gera.