ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
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Please submit all new problem proposals and their solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468, or by email at reuler@nwmissouri.edu. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by November 15, 2011. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2,\ \beta = (1-\sqrt{5})/2,\ F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1086 Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain.

Let n be a positive integer. Prove that

$$\frac{F_{2n+1} + F_nF_{n+1} + 1}{F_{n+2} + \displaystyle\sum_{1\le i<j\le n} F_iF_j}$$

is an integer and determine its value.

B-1087 Proposed by Br. J. Mahon, Kensington, Australia.

Evaluate

$$\prod_{i=3}^{\infty} \frac{F_{2i-1}^2 + F_{2i-1} - 2}{F_{2i-1}^2 - F_{2i-1} - 2}.$$

B-1088 Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain.

Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers defined by $a_1 = 3$, $a_2 = 5$ and for all $n \geq 3$, $a_{n+1} = \frac{1}{2}(a_n^2 + 1)$. Prove that

$$1 + \left(\sum_{k=1}^{n} \frac{L_k}{\sqrt{1 + a_k}} \right)^2 < \frac{L_n L_{n+1}}{2}.$$

B-1089 Proposed by Mohammad K. Azarian, University of Evansville, Indiana.

Let the sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ be defined by the recurrence relation

$$\frac{a_{n+2}}{a_n} = a_{n+1}, n \geq 0, a_0 = 1, a_1 = 2.$$

Also, let $p_1, p_2, p_3, \ldots, p_n$, represent a permutation of $a_1, a_2, a_3, \ldots, a_n$. Show that

$$\prod_{j=1}^{n!} \left(\sum_{i=1}^{n} |a_i - p_i| \right) a_j$$

is divisible by $2^{n!+F_{n!+2}-1}$ for $n \geq 0$.

B-1090 Proposed by N. Gauthier, Kingston, ON, Canada.

a) Let x be an arbitrary variable and k, m, n, r nonnegative integers, with $0 \leq k \leq n$ and $0 \leq r \leq m$. Also let $\{S_r^{(m)} : 0 \leq r \leq m\}$ be the set of Stirling numbers of the second kind, which satisfy the following recurrence, with initial value $S_0^{(0)} = 1$ and boundary conditions $S_{-1}^{(m)} = 0$ and $S_{m+1}^{(m)} = 0$:

$$S_r^{(m+1)} = r S_r^{(m)} + S_{r-1}^{(m)}.$$

Finally, let $(n)_r = n(n-1)\cdots(n-r+1)$ for $r \neq 0$, with $(n)_0 = 1$, and adopt the convention that $k^0 = 1$ for all values of k, including zero. Prove that:

$$\sum_{k=0}^{n} k^m \binom{n}{k} x^k = \sum_{r=0}^{m} (n)_r S_r^{(m)} x^r (1+x)^{n-r}.$$

b) Use a) to show that

$$\sum_{k=0}^{n} k^4 \binom{n}{k} F_k = a(n) F_{2n-3} + b(n) F_{2n-4}$$

and determine the polynomial coefficients, $a(n)$ and $b(n)$.

SOLUTIONS

A Sequence of Nested Radicals

B-1066 Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 48.2, May 2010)

Determine the value of

$$\sqrt{1+F_2\sqrt{1+F_4\sqrt{1+F_6\sqrt{\cdots\sqrt{1+F_{2n}\sqrt{\cdots}}}}}}.$$

Solution by Kenneth B. Davenport, Dallas, PA.

A solution to this intriguing relation depends on the identity

$$F_{2n}^2-1=F_{2n-2}F_{2n+2}. \tag{1}$$

This relation can be established via the known result appearing as I_{13} in *Fibonacci and Lucas Numbers*, by

$$F_{n-1}F_{n+1}-F_n^2=(-1)^n,\ \ n\geq 1. \tag{2}$$

Substituting $2n$ for n in (2) yields

$$F_{2n-1}F_{2n+1}-F_{2n}^2=1. \tag{3}$$

Now in the RHS of (1) we substitute $F_{2n}-F_{2n-1}$ for F_{2n-2}, and $F_{2n}+F_{2n+1}$ for F_{2n+2} to get

$$(F_{2n}-F_{2n-1})(F_{2n}+F_{2n+1})$$

or

$$F_{2n}^2-F_{2n}F_{2n-1}+F_{2n}F_{2n+1}-F_{2n-1}F_{2n+1}$$

which is

$$\begin{aligned}&F_{2n}^2+F_{2n}(F_{2n+1}-F_{2n-1})-F_{2n-1}F_{2n+1}\\&=F_{2n}^2+F_{2n}^2-F_{2n-1}F_{2n+1}.\end{aligned}$$

Using (2) and (3), the above simplifies to F_{2n}^2-1 and this proves relation (1). This implies $F_4=\sqrt{1+F_2F_6}$ and $F_6=\sqrt{1+F_4F_8}$. This substitution yields

$$F_4=\sqrt{1+F_2\sqrt{1+F_4F_8}}.$$

Continuing in this fashion by substituting $\sqrt{1+F_6F_{10}}$ for F_8, it is straightforward to derive an endless number of nested squareroots. This, then, establishes Ohtsuka's relation, i.e.

$$F_4=3=\sqrt{1+F_2\sqrt{1+F_4\sqrt{1+F_6\sqrt{\ldots\sqrt{1+F_{2n}\sqrt{\ldots}}}}}}.$$

Also solved by Paul S. Bruckman, Russell J. Hendel, and the proposer.

Close The Sum!

B-1067 Proposed by N. Gauthier, Royal Military College of Canada, Kingston, ON, Canada (Vol. 48.2, May 2010)

Let n be a positive integer. Find a closed-form expression for

$$\sum_{k=1}^{n} kF_k^3.$$

Solution by Charles K. Cook, Sumter, SC 29150

Differentiating $\sum_{k=1}^{n} x^k = \frac{x(1-x^n)}{1-x}$ and multiplying by x yields

$$\sum_{k=1}^{n} kx^k = \frac{x(1-x)[1-(n+1)x^n] + x^2(1-x^n)}{(1-x)^2}. \tag{1}$$

Next

$$F_k^3 = \left(\frac{\alpha^k - \beta^k}{\sqrt{5}}\right)^3 = \frac{\alpha^{3k} - \beta^{3k} - 3(-1)^k(\alpha^k - \beta^k)}{5\sqrt{5}}. \tag{2}$$

Using $\alpha\beta = -1$, $(1-\alpha^3)^2 = 4\alpha^2$, $(1-\beta^3)^2 = 4\beta^2$, $1+\alpha = \alpha^2$, and $1+\beta = \beta^2$ as needed, it follows from (1) that

$$A = \sum_{k=1}^{n} k\alpha^{3k} = \frac{-2\alpha^2[1-(n+1)\alpha^{3n}] + \alpha^4(1-\alpha^{3n})}{4}$$

$$B = \sum_{k=1}^{n} k\beta^{3k} = \frac{-2\beta^2[1-(n+1)\beta^{3n}] + \beta^4(1-\beta^{3n})}{4}$$

$$C = \sum_{k=1}^{n} k(-\alpha)^k = \frac{-\alpha + (n+1)(-1)^n\alpha^{n+1} + 1 - (-1)^n\alpha^n}{\alpha^2}$$

and

$$D = \sum_{k=1}^{n} k(-\beta)^k = \frac{-\beta + (n+1)(-1)^n\beta^{n+1} + 1 - (-1)^n\beta^n}{\beta^2}.$$

Subtracting yields

$$\begin{aligned} A - B &= \frac{-2(\alpha^2-\beta^2) + 2(n+1)(\alpha^{3n+2}-\beta^{3n+2}) + (\alpha^4-\beta^4) - (\alpha^{3n+4}-\beta^{3n+4})}{4} \\ &= \frac{\sqrt{5}}{4}[1 + 2(n+1)F_{3n+2} - F_{3n+4}] \end{aligned}$$

and

$$\begin{aligned} C - D &= -(\alpha-\beta) - (\alpha^2-\beta^2) + (n+1)(-1)^n(\alpha^{n-1}-\beta^{n-1}) - (-1)^n(\alpha^{n-2}-\beta^{n-2}) \\ &= -\sqrt{5}[2 - (n+1)(-1)^nF_{n-1} + (-1)^nF_{n-2}]. \end{aligned}$$

Substituting these expressions into (2) and simplifying yields

$$\sum_{k=1}^{n} kF_k^3 = \frac{25 + 2nF_{3n+2} - F_{3n+1} - 12(-1)^n(nF_{n-1} - F_{n-3})}{20}.$$

Also solved by Paul S. Bruckman, Kenneth B. Davenport, G. C. Greubel, Russell J. Hendel, Jaroslav Seibert, David Terr, and the proposer.

Limit of a Sequence

B-1068 Proposed by Mohammad K. Azarian, University of Evansville, Indiana (Vol. 48.2, May 2010)

If the sequence $s_0, s_1, s_2, s_3, \ldots$ is defined by the difference equation

$$2ks_{k+1} + (1-2k)s_k - s_{k-1} = 0, (k \geq 1), s_0 = F_n, s_1 = F_{2n},$$

then write $\lim_{k\to\infty} s_k$ in terms of Lucas numbers.

Solution by Francisco Perdomo and Ángel Plaza (jointly), Universidad de Las Palmas de Gran Canaria, Las Palmas G. C., Spain

The difference equation defining s_k may be written for $k > 0$, as follows

$$s_{k+1} - s_k = \frac{-1}{2k}(s_k - s_{k-1}).$$

It follows that $s_{k+1} - s_k = \left(\frac{-1}{2}\right)^k \frac{1}{k!}(s_1 - s_0)$ and so

$$\begin{aligned}
\lim_{k\to\infty} s_k &= s_0 + (s_1 - s_0) + (s_2 - s_1) + (s_3 - s_2) + \ldots \\
&= s_0 + (s_1 - s_0) \cdot \sum_{k=0}^{\infty} \left(\frac{-1}{2}\right)^k \frac{1}{k!} \\
&= s_0 + (s_1 - s_0) \cdot e^{-1/2} \\
&= F_n + \frac{F_{2n} - F_n}{\sqrt{e}} \\
&= \frac{L_{n-1} + L_{n+1}}{5}\left(1 + \frac{L_n - 1}{\sqrt{e}}\right).
\end{aligned}$$

Also solved by Paul S. Bruckman and the proposer.

A Fibonacci "Dis–Array"!

B-1069 Proposed by Pantelimon George Popescu, Politechnica University, Bucharest, Romania and José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain (Vol. 48.2, May 2010)

Let n, a, b, c, d be positive integers and A be a square matrix for which

$$A^n = \begin{pmatrix} F_{a+n-1} & L_{b+n-1} \\ L_{c+n-1} & F_{d+n-1} \end{pmatrix}.$$

Show that $b = c$ and $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ or $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

Solution by Steve Edwards, Southern Polytechnic State University, Marietta, GA

Correction: The matrix should consist of all Fibonacci numbers. Also, a and d should be non-negative rather than positive.
Since $A = \begin{pmatrix} F_a & F_b \\ F_c & F_d \end{pmatrix}$, $A^2 = \begin{pmatrix} F_{a+1} & F_{b+1} \\ F_{c+1} & F_{d+1} \end{pmatrix}$, but also,

$$A^2 = AA = \begin{pmatrix} F_a^2 + F_bF_c & F_aF_b + F_bF_d \\ F_aF_c + F_cF_d & F_bF_c + F_d^2 \end{pmatrix},$$

we have $F_{b+1} = F_b(F_a + F_d)$ and $F_{c+1} = F_c(F_a + F_d)$, from which it follows that $b = c$. But since $\frac{F_{b+1}}{F_b} = F_a + F_d$, $\frac{F_{b+1}}{F_b}$ is an integer, which is only possible for $b = 1$ or 2.

For $b = 2$, $F_a + F_d = 2$, but this implies that F_a and F_d have values 0, 1, or 2, all of which lead to contradictions.

For $b = 1$, $\frac{F_{b+1}}{F_b} = \frac{F_2}{F_1} = 1 = F_a + F_d$, so either $F_a = 1$ and $F_d = 0$ or vice versa. Both F_1 and F_2 equal 1, but it is easily seen that we need $a = 2$ and $d = 0$ or $a = 0$ and $d = 2$ in order for the equation for A^n to be satisfied, i.e. $A = \begin{pmatrix} F_2 & F_1 \\ F_1 & F_0 \end{pmatrix}$ or $A = \begin{pmatrix} F_0 & F_1 \\ F_1 & F_2 \end{pmatrix}$. Finally, note that for $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, the equation with A^n is satisfied since

$$A^{k+1} = A\begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix} = \begin{pmatrix} F_{k+1} + F_k & F_k + F_{k-1} \\ F_k & F_{k-1} \end{pmatrix} = \begin{pmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{pmatrix}.$$

The case $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ is similar.

A corrected version was also solved by Paul S. Bruckman and Russell J Hendel.

A Trigonometric Identity

B-1070 Proposed by Roman Witula and Damian Slota, Silesian University of Technology, Poland.
(Vol. 48.2, May 2010)

Prove that

$$\left|\tan\left(x + \frac{1}{2}\arctan 2\right)\right| \cdot \left|\frac{\alpha - \tan x}{\beta - \tan x}\right| \equiv \alpha,$$

for all $x \neq (2k-1)\frac{\pi}{2}$, $x \neq k\pi + \arctan\beta$, $x \neq k\pi + \arctan\alpha$, $x \neq k\frac{\pi}{2} - \frac{1}{2}\arctan 2$ and $k \in \mathbb{Z}$.

Solution by Paul S. Bruckman, 38 Front St., Unit #302, Nanaimo, BC, V9R 0B8, Canada

For the moment, we assume that x is such that the expression in the statement of the problem is well-defined. Now $\tan^{-1} 2 = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$, so

$$\frac{1}{2}\tan^{-1} 2 = \tan^{-1}\sqrt{\frac{1-\frac{1}{\sqrt{5}}}{1+\frac{1}{\sqrt{5}}}} = \tan^{-1}\sqrt{\frac{\sqrt{5}-1}{\sqrt{5}+1}} = \tan^{-1}\frac{\sqrt{5}-1}{2} = \tan^{-1}\left(\frac{1}{\alpha}\right).$$

Therefore,

$$\tan\left(x+\frac{1}{2}\arctan 2\right) = \frac{\tan x + \frac{1}{\alpha}}{1-\frac{\tan x}{\alpha}} = \frac{\alpha\tan x + 1}{\alpha - \tan x}.$$

Then,

$$\tan\left(x+\frac{1}{2}\arctan 2\right)\cdot\frac{\alpha-\tan x}{\beta-\tan x} = \frac{\alpha\tan x+1}{\beta-\tan x} = \frac{\alpha(\tan x-\beta)}{\beta-\tan x} = -\alpha;$$

therefore,

$$\left|\tan\left(x+\frac{1}{2}\arctan 2\right)\cdot\frac{\alpha-\tan x}{\beta-\tan x}\right| = \alpha.$$

We must exclude the possible values of x that might be singularities. First of all, $\tan x$ must be well-defined, which implies that $x \neq (2k-1)\frac{\pi}{2}$ when $k \in \mathbb{Z}$. Secondly, $\alpha - \tan x$ and $\beta - \tan x$ must not vanish; that is, $x \neq k\pi + \arctan\alpha$ and $x \neq k\pi + \arctan\beta$. Finally, $\tan(x+\frac{1}{2}\arctan 2)$ must be well-defined, so $x \neq (2k-1)\frac{pi}{2} - (\frac{1}{2}\arctan 2)$, or $x \neq (2k-1\frac{pi}{2} + \arctan\beta$. We may combine two of these conditions as follows: $x \neq k\frac{pi}{2} + \arctan\beta = k\frac{pi}{2} - (\frac{1}{2}\arctan 2)$.

Also solved by Charles K. Cook, Kenneth B. Davenport, Steve Edwards, G. C. Greubel, Russell J. Hendel, Zbigniew Jakubczyk, Jaroslav Seibert, David Terr, and the proposers.