ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
RUSS EULER AND JAWAD SADEK

Please submit all new problem proposals and their solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468, or by email at reuler@nwmissouri.edu. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by May 15, 2013. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2,\ \beta = (1-\sqrt{5})/2,\ F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1116 Proposed by M. N. Deshpande, Nagpur, India

Let n be a nonnegative integer and let T_n be the nth triangular number. Prove that each of the following is divisible by 5:

$$nL_{n+1} + 2F_n \quad (1)$$

$$T_nL_{n+1} + (n+1)F_n \quad (2)$$

B-1117 Proposed by D. M. Bătineţu–Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" Secondary School, Buzău, Romania

Prove that:

$$\sum_{k=1}^{n}\left(\sqrt{F_k^4-F_k^2+1}+\frac{F_k^2-1}{F_k^4+1}\right)<F_nF_{n+1} \tag{1}$$

$$\sum_{k=1}^{n}\left(\sqrt{L_k^4-L_k^2+1}+\frac{L_k^2-1}{L_k^4+1}\right)<L_nL_{n+1}-2 \tag{2}$$

for any positive integer n.

B-1118 Proposed by Gordon Clarke, Brisbane, Australia

If n is a nonnegative integer, prove that:

$$\left(F_n^2+F_{n+1}^2\right)\left(F_{n+2}^2+F_{n+3}^2\right)=F_{2n+3}^2+1 \tag{1}$$

$$\left(F_n^2+F_{n+2}^2\right)\left(F_{n+4}^2+F_{n+6}^2\right)=F_{2n+6}^2+\left(2F_{n+3}^2\pm 5\right)^2. \tag{2}$$

B-1119 Proposed by D. M. Bătineţu–Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" Secondary School, Buzău, Romania

Prove that

$$\sum_{k=1}^{n}\frac{\tan^2\frac{x}{2^k}}{2^kF_k^2}\geq\frac{\left(\cot\frac{x}{2^n}-2^{n+1}\cot 2x\right)^2}{2^{2n}F_nF_{n+1}},\ \text{for any } x\in\left(0,\frac{\pi}{4}\right).$$

B-1120 Proposed by the Problem Editor

Prove or disprove: $F_n\equiv 2n3^n \pmod 5$ for all nonnegative integers n.

SOLUTIONS

An Identity Involving Sums of Ratios of Fibonacci and Lucas Numbers

B-1096 Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 49.4, November 2011)

For $n \geq 2$, prove that

$$\left(\sum_{k=1}^{n-1} \frac{L_k}{F_k}\right)\left(\sum_{k=1}^{n-1} \frac{1}{L_k L_{n-k}}\right) = \left(\sum_{k=1}^{n-1} \frac{F_k}{L_k}\right)\left(\sum_{k=1}^{n-1} \frac{1}{F_k F_{n-k}}\right).$$

Solution Harris Kwong, SUNY Fredonia, Fredonia, NY

Using Binet's formulas, it is easy to verify that $F_k L_{n-k} + F_{n-k} L_k = 2F_n$. Hence,

$$\frac{1}{L_k L_{n-k}} = \frac{1}{2F_n}\left(\frac{F_k L_{n-k} + F_{n-k} L_k}{L_k L_{n-k}}\right) = \frac{1}{2F_n}\left(\frac{F_k}{L_k} + \frac{F_{n-k}}{L_{n-k}}\right),$$

$$\frac{1}{F_k F_{n-k}} = \frac{1}{2F_n}\left(\frac{F_k L_{n-k} + F_{n-k} L_k}{F_k F_{n-k}}\right) = \frac{1}{2F_n}\left(\frac{L_{n-k}}{F_{n-k}} + \frac{L_k}{F_k}\right),$$

and

$$\sum_{k=1}^{n-1} \frac{1}{L_k L_{n-k}} = \frac{1}{2F_n}\sum_{k=1}^{n-1}\left(\frac{F_k}{L_k} + \frac{F_{n-k}}{L_{n-k}}\right) = \frac{1}{F_n}\sum_{k=1}^{n-1}\frac{F_k}{L_k},$$

$$\sum_{k=1}^{n-1} \frac{1}{F_k F_{n-k}} = \frac{1}{2F_n}\sum_{k=1}^{n-1}\left(\frac{L_{n-k}}{F_{n-k}} + \frac{L_k}{F_k}\right) = \frac{1}{F_n}\sum_{k=1}^{n-1}\frac{L_k}{F_k},$$

from which the desired result follows immediately.

Also solved by Paul S. Bruckman, G. C. Greubel, and the proposer.

An Infinite Arctangent Series

B-1097 Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain (Vol. 49.4, November 2011)

Compute the following sum

$$\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{L_{n-1}(1+F_nF_{n+1}) - F_{n-1}(1+L_nL_{n+1})}{F_{2n-2} + (1+L_nL_{n+1})(1+F_nF_{n+1})}\right).$$

Solution by Paul S. Bruckman, Nanaimo, BC, Canada

Let

$$\rho_n = \tan^{-1}\left(\frac{1}{F_{n+1}}\right), n \geq 1;$$

then $\pi/2 - \rho_{n-1} = \tan^{-1}(F_n)$. Hence,

$$\pi/2 - \rho_{n-1} + \rho_n = \tan^{-1}\left\{\frac{\frac{1}{F_{n+1}} + F_n}{1 - \frac{F_n}{F_{n+1}}}\right\} = \tan^{-1}\left\{\frac{1 + F_nF_{n+1}}{F_{n+1} - F_n}\right\} = \tan^{-1}\left\{\frac{1 + F_nF_{n+1}}{F_{n-1}}\right\}.$$

Likewise, if $\sigma_n = \tan^{-1}\left(\frac{1}{L_{n+1}}\right)$; then $\pi/2 - \sigma_{n-1} = \tan^{-1}(L_n)$, and

$$\pi/2 - \sigma_{n-1} + \sigma_n = \tan^{-1}\left\{\frac{1 + L_nL_{n+1}}{L_{n-1}}\right\}.$$

Then

$$\begin{aligned}-\rho_{n-1} + \rho_n + \sigma_{n-1} - \sigma_n &= \tan^{-1}\left\{\frac{\frac{1+F_nF_{n+1}}{F_{n-1}} - \left(\frac{1+L_nL_{n+1}}{L_{n-1}}\right)}{1 + \frac{(1+F_nF_{n+1})(1+L_nL_{n+1})}{F_{n-1}L_{n-1}}}\right\} \\ &= \tan^{-1}\left\{\frac{L_{n-1}(1 + F_nF_{n+1}) - F_{n-1}(1 + L_nL_{n+1})}{F_{2n-2} + (1 + L_nL_{n+1})(1 + F_nF_{n+1})}\right\}.\end{aligned}$$

Therefore, the given sum is equal to

$$\sum_{n=1}^{\infty}(-\rho_{n-1}+\rho_n+\sigma_{n-1}-\sigma_n) = \sigma_0-\rho_0 = \tan^{-1}\left(\frac{1}{L_1}\right)-\tan^{-1}\left(\frac{1}{F_1}\right) = \tan^{-1}(1)-\tan^{-1}(1) = 0.$$

Also solved by Brian D. Beasley, Kenneth B. Davenport, Amos E. Gera, G. C. Greubel, Ahmad Habil, Ángel Plaza, Jaroslav Seibert, and the proposer.

By Means of the Geometric and Arithmetic Means

B-1098 Proposed by Sergio Falcón and Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain (Vol. 49.4, November 2011)

Let n be a positive integer. Prove that

$$\left(\sum_{k=1}^{n}\frac{L_k^2}{\sqrt{1+L_k^2}}\right)\left(\prod_{k=1}^{n}(1+L_k^2)\right)^{1/2n} \le L_nL_{n+1} - 2.$$

Solution by Shanon Michaels Areford, University of South Carolina Sumter and André Galinda, student at Universidad Distrial Francisco José de Caldas, Bogotá, Columbia (separately).

Using that the Geometric Mean is less than or equal to the Arithmetic Mean, we can see that

$$\left(\sum_{k=1}^{n}\frac{L_k^2}{\sqrt{1+L_k^2}}\right)\left(\prod_{k=1}^{n}(1+L_k^2)\right)^{1/2n} \le \frac{\left(\sum_{k=1}^{n}\frac{L_k^2}{\sqrt{1+L_k^2}}\right)\left(\sum_{k=1}^{n}\sqrt{1+L_k^2}\right)}{n}.$$

Since $\left\{\frac{L_k^2}{\sqrt{1+L_k^2}}\right\}$ and $\left\{\sqrt{1+L_k^2}\right\}$ are non-decreasing sequences, we conclude that

$$\frac{\left(\sum_{k=1}^n \frac{L_k^2}{\sqrt{1+L_k^2}}\right)\left(\sum_{k=1}^n \sqrt{1+L_k^2}\right)}{n} \leq \sum_{k=1}^n \frac{L_k^2}{\sqrt{1+L_k^2}}\sqrt{1+L_k^2} = \sum_{k=1}^n L_k^2.$$

Since $\sum_{k=1}^n L_k^2 = L_n L_{n+1} - 2$, see [1, page 78], the inequality follows.

References

[1] T. Koshy, *Fibonacci and Lucas Numbers and Applications*, John Wiley & Sons, Inc., New York, 2001.

Also solved by Paul S. Bruckman, Kenneth B. Davenport, G. C. Greubel, Russell Jay Hendel, Zbigniew Jakubczyk, and the proposer.

On k-Fibonacci and k-Lucas Numbers

B-1099 **Proposed by Sergio Falcón and Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain**
(Vol. 49.4, November 2011)

For any positive integer k, the k-Fibonacci and k-Lucas sequences, $\{F_{k,n}\}_{n\in\mathbb{N}}$ and $\{L_{k,n}\}_{n\in\mathbb{N}}$, both are defined recursively by $u_{n+1} = ku_n + u_{n-1}$ for $n \geq 1$, with respective initial conditions $F_{k,0} = 0$; $F_{k,1} = 1$ and $L_{k,0} = 2$; $L_{k,1} = k$. Prove that

$$2^{n-1}F_{k,n} = \sum_{i\geq 0} k^{n-1-2i}(k^2+4)^i \binom{n}{2i+1}. \tag{1}$$

$$2^{n-1}L_{k,n} = \sum_{i\geq 0} k^{n-2i}(k^2+4)^i \binom{n}{2i}. \tag{2}$$

$$2^{n+1}F_{k,n+1} = \sum_{i=0}^{n} k^{n-i}2^i L_{k,i}. \tag{3}$$

Solution by Russell J. Hendel, Towson University, Towson, MD.

We suffice with the proof of problem identity (2), the proof of the other problem identities being similar.

The k-Fibonacci and k-Lucas numbers have Binet forms, [2], defined by

$$D = k^2+4, \alpha_k = \frac{k+\sqrt{D}}{2}, \beta_k = \frac{k-\sqrt{D}}{2}, F_{k,n} = \frac{\alpha_k^n - \beta_k^n}{\sqrt{D}}, L_{k,n} = \alpha_k^n + \beta_k^n.$$

We follow the technique described in [1, pp. 167-168]. By the binomial theorem, it follows from the Binet forms that

$$2^n L_{k,n} = (k+\sqrt{D})^n + (k-\sqrt{D})^n = \sum_{i\geq 0}\binom{n}{i}k^{n-i}D^{i/2}(1+(-1)^i) = 2\sum_{i\geq 0}\binom{n}{2i}k^{n-2i}D.$$

Problem identity (2) now immediately follows from the defining relation $D = k^2+4$.

An almost identical proof — rewriting the k-Fibonacci and the k-Lucas numbers using their binet form, applying the binomial theorem to the definitions of α_k and β_k, and combining like terms — proves the other problem identities.

References

[1] T. Koshy, *Fibonacci and Lucas Numbers and Applications*, John Wiley & Sons, Inc., New York, 2001.
[2] Wolfram Website, `http://mathworld.wolfram.com/LucasSequence.html`, *Lucas Sequence*.

Also solved by Paul S. Bruckman, Kenneth B. Davenport, G. C. Greubel, Jaroslav Seibert, and the proposer.

More of k-Fibonacci and k-Lucas Numbers

B-1100 Proposed by Sergio Falcón and Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.
(Vol. 49.4, November 2011)

For any positive integer k, the k-Fibonacci and k-Lucas sequences, $\{F_{k,n}\}_{n\in\mathbb{N}}$ and $\{L_{k,n}\}_{n\in\mathbb{N}}$, both are defined recursively by $u_{n+1} = ku_n + u_{n-1}$ for $n \geq 1$, with respective initial conditions $F_{k,0} = 0$; $F_{k,1} = 1$, and $L_{k,0} = 2$; $L_{k,1} = k$. Prove that

$$\sum_{i\geq 0} \binom{2n}{i} F_{k,2i+1} = (k^2+4)^n F_{k,2n+1}. \tag{1}$$

$$\sum_{i\geq 0} \binom{2n+1}{i} F_{k,2i} = (k^2+4)^{n+1} L_{k,2n+1}. \tag{2}$$

$$\sum_{i\geq 0} \binom{2n}{i} L_{k,2i} = (k^2+4)^n L_{k,2n}. \tag{3}$$

$$\sum_{i\geq 0} \binom{2n+1}{i} L_{k,2i} = (k^2+4)^{n+1} F_{k,2n+1}. \tag{4}$$

Solution by Paul S. Bruckman, Nanaimo, BC, Canada

We may readily show the following:

(a) $F_{k,n} = \frac{\rho^n - \sigma^n}{\rho - \sigma}$, $L_{k,n} = \rho^n + \sigma^n$, $n = 0, 1, 2, \ldots$, where
(b) $\rho = \frac{k+\sqrt{k^2+4}}{2}$, $\sigma = \frac{k-\sqrt{k^2+4}}{2}$; note $\rho + \sigma = k$, $\rho - \sigma = \sqrt{k^2+4}$, $\rho\sigma = -1$.

Proof of Part (1):

$$\begin{aligned}
\sum_{i\geq 0}\binom{2n}{i}F_{k,2i+1} &= \frac{1}{\rho-\sigma}\sum_{i=0}^{2n}\binom{2n}{i}(\rho^{2i+1}-\sigma^{2i+1}) \\
&= \frac{\rho}{\rho-\sigma}(1+\rho^2)^{2n} - \frac{\sigma}{\rho-\sigma}(1+\sigma^2)^{2n} \\
&= \rho^{2n+1}(\rho-\sigma)^{2n-1} - \sigma^{2n+1}(\rho-\sigma)^{2n-1} \\
&= (\rho-\sigma)^{2n}\left\{\frac{\rho^{2n+1}-\sigma^{2n+1}}{\rho-\sigma}\right\} \\
&= (k^2+4)^n F_{k,2n+1}.
\end{aligned}$$

Proof of Part (2) (answer modified):

$$\begin{aligned}
\sum_{i\geq 0}\binom{2n+1}{i}F_{k,2i} &= \frac{1}{\rho-\sigma}\sum_{i=0}^{2n+1}\binom{2n+1}{i}(\rho^{2i}-\sigma^{2i}) \\
&= \frac{\rho}{\rho-\sigma}(1+\rho^2)^{2n+1} - \frac{1}{\rho-\sigma}(1+\sigma^2)^{2n+1} \\
&= \rho^{2n+1}(\rho-\sigma)^{2n} - \sigma^{2n+1}(\rho-\sigma)^{2n} \\
&= (\rho-\sigma)^{2n}\left\{\rho^{2n+1}+\sigma^{2n+1}\right\} \\
&= (k^2+4)^n L_{k,2n+1}.
\end{aligned}$$

Note that this answer differs from that given in the statement of the problem.

Proof of Part (3):

$$\begin{aligned}
\sum_{i\geq 0}\binom{2n}{i}L_{k,2i} &= \sum_{i=0}^{2n}\binom{2n}{i}(\rho^{2i}+\sigma^{2i}) \\
&= (1+\rho^2)^{2n} + (1+\sigma^2)^{2n} \\
&= \rho^{2n}(\rho-\sigma)^{2n} + \sigma^{2n}(\rho-\sigma)^{2n} \\
&= (\rho-\sigma)^{2n}\left\{\rho^{2n}+\sigma^{2n}\right\} \\
&= (k^2+4)^n L_{k,2n}.
\end{aligned}$$

Proof of Part (4):

$$\begin{aligned}
\sum_{i\geq 0}\binom{2n+1}{i}L_{k,2i} &= \sum_{i=0}^{2n+1}\binom{2n+1}{i}(\rho^{2i}+\sigma^{2i}) \\
&= (1+\rho^2)^{2n+1} + (1+\sigma^2)^{2n+1} \\
&= \rho^{2n+1}(\rho-\sigma)^{2n+1} - \sigma^{2n+1}(\rho-\sigma)^{2n+1} \\
&= (\rho-\sigma)^{2n+2}\left\{\frac{\rho^{2n+1}-\sigma^{2n+1}}{\rho-\sigma}\right\} \\
&= (k^2+4)^{n+1} F_{k,2n+1}.
\end{aligned}$$

Also solved by Kenneth B. Davenport, Amos E. Gera, G. C. Greubel, Russell J. Hendel, Zbigniew Jakubczyk, Jaroslav Seibert, and the proposer.

We would like to belatedly acknowledge Amos E. Gera for the solution of Problems B-1091 and B-1093.