ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
RUSS EULER AND JAWAD SADEK

Please submit all new problem proposals and their solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468, or by email at reuler@nwmissouri.edu. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by August 15, 2014. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2,\ \beta = (1-\sqrt{5})/2,\ F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1141 **Proposed by Hideyuki Ohtsuka, Saitama, Japan.**

Determine

$$\sum_{k=1}^{\infty} \frac{2^k \sin(2^k\theta)}{L_{2^k} + 2\cos(2^k\theta)}.$$

B-1142 **Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade School, Buzău, Romania.**

Prove that $\sum_{k=1}^{n} F_{4k-1} = F_{2n} \cdot F_{2n+1}$ for any positive integer n.

B-1143 **Proposed by José Luis Díaz-Barrero, BARCELONA TECH, Barcelona, Spain and Francesc Gispert Sánchez, CFIS, BARCELONA TECH, Barcelona, Spain.**

Let n be a positive integer. Prove that

$$\frac{1}{F_n F_{n+1}} \left[\left(1 - \frac{1}{n}\right) \sum_{k=1}^{n} F_k^{2n} + \prod_{k=1}^{n} F_k^2 \right] \geq \left(\prod_{k=1}^{n} F_k^{(1-1/n)} \right)^2 .$$

B-1144 **Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade School, Buzău, Romania.**

Prove that

$$\prod_{k=1}^{n} (F_k^2 + 1) > F_n \cdot F_{n+1} + 1 \tag{1}$$

$$\prod_{k=1}^{n} (L_k^2 + 1) > L_n \cdot L_{n+1} - 1 \tag{2}$$

for any positive integer n.

B-1145 **Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade School, Buzău, Romania.**

Prove that

$$\left(F_1 - \sqrt{F_1 F_2} + F_2\right)^2 + \left(F_2 - \sqrt{F_2 F_3} + F_3\right)^2 + \cdots + \left(F_n - \sqrt{F_n F_1} + F_1\right)^2 \geq F_n F_{n+1} \tag{1}$$

$$\left(L_1 - \sqrt{L_1 L_2} + L_2\right)^2 + \left(L_2 - \sqrt{L_2 L_3} + L_3\right)^2 + \cdots + \left(L_n - \sqrt{L_n L_1} + L_1\right)^2 \geq L_n L_{n+1} - 2 \tag{2}$$

for any positive integer n.

SOLUTIONS

Inequalities With 4's

B-1121 Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade General School, Buzău, Romania (Vol. 51.1, February 2013)

Prove that

$$n+4+4F_nF_{n+1} > 4F_{n+2} \tag{1}$$

and

$$n+4+4L_nL_{n+1} > 4L_{n+2} \tag{2}$$

for any positive integer n.

Solution by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

Note that inequality (1) may be written as

$$\frac{n}{4} > F_{n+2} - 1 - F_nF_{n+1}.$$

Now, since $F_{n+2}-1 = \sum_{k=1}^{n} F_k$ and $F_nF_{n+1} = \sum_{k=1}^{n} F_k^2$, the conclusion follows trivially.

A similar argument may be applied to inequality (2), since $L_{n+2} - 3 = \sum_{k=1}^{n} L_k$, and $L_nL_{n+1} - 2 = \sum_{k=1}^{n} L_k^2$.

Also solved by Gurdial Arora and Sindhu Unnithan (jointly), Brian D. Beasly, Paul S. Bruckman, Charles K. Cook, Dmitry Fleishman, Amos E. Gera, Russell J. Hendel, Charles McCraken, Jaroslav Seibert, David Stone and John Hawkins (jointly), and the proposer.

An Odd Mod

B-1122 Proposed by Harris Kwong, SUNY Fredonia, Fredonia, NY (Vol. 51.1, February 2013)

Prove that, given any integer $r \geq 4$ if $\gcd(2r-1, r^2-r-1) = 1$, then

$$F_{n+\phi(r^2-r-1)} \equiv F_n, \quad (\mathrm{mod}\ r^2-r-1)$$

for all nonnegative integers n. Here, ϕ denotes Euler's phi-function.

Solution by the proposer.

From

$$\begin{aligned}
\sum_{n\geq 0} F_n x^n &= \frac{x}{1-x-x^2} \\
&\equiv \frac{x}{(1-rx)[1-(1-r)x]} \\
&\equiv (2r-1)^{-1}\left(\frac{1}{1-rx} - \frac{1}{1-(1-r)x}\right) \quad (\mathrm{mod}\ r^2-r-1).
\end{aligned}$$

Therefore,

$$F_n \equiv (2r-1)^{-1}[r^n + (1-r)^n] \pmod{r^2 - r - 1}.$$

Since $r(1-r) \equiv -1 \pmod{r^2-r-1}$, it is clear that the inverses of r and $1-r$ exist. Hence, their orders divide $\phi(r^r - r - 1)$. Consequently,

$$r^n \equiv r^{n+\phi(r^2-r-1)} \pmod{r^2-r-1}$$

and

$$(1-r)^n \equiv (1-r)^{n+\phi(r^2-r-1)} \pmod{r^2-r-1}.$$

This result follows immediately.

Also solved by Paul S. Bruckman.

Square Roots and Cubes of Fibonacci Numbers

B-1123 Proposed by José Luis Díaz-Barrero, BARCELONA TECH, Barcelona, Spain and Mihály Bencze, Braşov, Romania. (Vol. 50.1, February 2012)

Let $n \geq 2$ be a positive integer. Prove that

$$\frac{1}{n}\sum_{k=1}^{n} \frac{F_k^3}{F_nF_{n+1} - F_k^2} \geq \frac{1}{n-1}\sqrt{\frac{1}{F_{n+2}-1}\sum_{k=1}^{n} F_k^3}.$$

Solution by Paul S. Bruckman.

We may express the given inequality as follows:

$$\frac{1}{n}\sum_{k=1}^{n} \frac{F_k^3}{\sum_{k=1}^{n} F_k^2 - F_k^2} \geq \frac{1}{n-1}\sqrt{\frac{\sum_{k=1}^{n} F_k^3}{\sum_{k=1}^{n} F_k}}, n = 2, 3, \ldots. \tag{1}$$

We use the following inequality: $\left(\frac{1}{n}\sum_{k=1}^{n} x_k^3\right)^{1/3} \geq \frac{1}{n}\sum_{k=1}^{n} x_k$, valid for positive x_k's, which reduces to

$$\sqrt{\frac{\sum_{k=1}^{n} x_k^3}{\sum_{k=1}^{n} x_k}} \geq \frac{1}{n}\sum_{k=1}^{n} x_k.$$

In particular, it suffices to show that

$$\frac{1}{n}\sum_{k=1}^{n} \frac{F_k^3}{\sum_{k=1}^{n} F_k^2 - F_k^2} \geq \frac{1}{(n-1)n}\sum_{k=1}^{n} F_k, n = 2, 3, \ldots. \tag{2}$$

The denominator of the expression on the left side of (2) satisfies

$$\sum_{k=1}^{n} F_k^2 - F_k^2 \geq \sum_{k=1}^{n-1} F_k^2.$$

It therefore suffices to prove that

$$\frac{1}{(n-1)}\sum_{k=1}^{n} F_k \leq \sum_{k=1}^{n} \frac{F_k^3}{\sum_{k=1}^{n-1} F_k^2} = \frac{\sum_{k=1}^{n} F_k^3}{\sum_{k=1}^{n-1} F_k^2}.$$

Equivalently, it suffices to prove the following:

$$\frac{1}{(n-1)}\sum_{k=1}^{n-1} F_k^2 \le \frac{\sum_{k=1}^{n} F_k^3}{\sum_{k=1}^{n} F_k}. \tag{3}$$

A stronger inequality actually holds, namely

$$\frac{1}{n}\sum_{k=1}^{n} F_k^2 \le \frac{\sum_{k=1}^{n} F_k^3}{\sum_{k=1}^{n} F_k}. \tag{4}$$

Equation (4) is stronger than equation (3) because its left member represents an average value of F_k^2 including the final term in the sum, which does not appear in (3); therefore,

$$\frac{1}{(n-1)}\sum_{k=1}^{n-1} F_k^2 \le \frac{1}{n}\sum_{k=1}^{n} F_k^2.$$

The right member of (4) also represents a weighted average value of F_k^2. However, the "weights" in the right member of (4) are the F_k's (an increasing sequence, from some point on); the weights in the left member are all equal to 1. This tends to shift the average to a higher value in the right member of (4), as opposed to its left member. This proves (4) and the original inequality.

Also solved by Dmitry Fleishman, Ángel Plaza, and the proposer.

Greater Than Half the Number of Terms

B-1124 Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade General School, Buzău, Romania. (Vol. 51.1, February 2013)

Prove that

$$\sum_{k=1}^{n}\left(\frac{F_k}{F_{k+3}} + \frac{F_{k+1}}{2F_k + F_{k+1}}\right) > \frac{n}{2} \tag{1}$$

$$\sum_{k=1}^{n}\left(\frac{L_k}{L_{k+3}} + \frac{L_{k+1}}{2L_k + L_{k+1}}\right) > \frac{n}{2} \tag{2}$$

for any positive integer n.

Solution by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

We know that $F_{k+3} = F_{k+2} + F_{k+1} = 2F_{k+1} + F_k$, and $L_{k+3} = 2L_{k+1} + L_k$. Hence, both inequalities may be proved similarly. We'll show (1) only.

$$\begin{aligned}\sum_{k=1}^{n}\left(\frac{F_k}{F_{k+3}}+\frac{F_{k+1}}{2F_k+F_{k+1}}\right) &= \sum_{k=1}^{n}\left(\frac{F_k}{F_k+2F_{k+1}}+\frac{F_{k+1}}{2F_k+F_{k+1}}\right)\\ &> \sum_{k=1}^{n}\left(\frac{F_k}{2F_k+2F_{k+1}}+\frac{F_{k+1}}{2F_k+2F_{k+1}}\right)\\ &= \frac{n}{2}.\end{aligned}$$

□

Also solved by Gurdial Arora and Sindhu Unnithan (jointly), Paul S. Bruckman, Charles K. Cook, Kenneth B. Davenport, Dmitry Fleishman, Amos E. Gera, Russell J. Hendel, Jaraslav Seibert, and the proposer.

A Lucas Sum

B-1125 Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade General School, Buzău, Romania.
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Prove that

$$\frac{(L_1^2+1)(L_2^2+1)}{L_1L_2+1}+\frac{(L_2^2+1)(L_3^2+1)}{L_2L_3+1}+\cdots+\frac{(L_{n-1}^2+1)(L_n^2+1)}{L_{n-1}L_n+1}+\frac{(L_n^2+1)(L_1^2+1)}{L_nL_1+1}\geq 2L_{n+2}-6,$$

for any positive integer n.

Solution by Kenneth B. Davenport, Dallas, PA.

Omitting the last term of the left-hand side, the inequality remains valid for all integers $n \geq 2$. The left-hand side then becomes

$$\sum_{k=1}^{n}\frac{(L_{k-1}^2+1)(L_k^2+1)}{(L_{k-1}L_k+1)}. \tag{1}$$

The right-hand side

$$2(L_{n+2}-3)=2\sum_{k=1}^{n}L_k \tag{2}$$

as shown in [1, p. 54].

Hence, we only need to show that

$$\frac{(L_{k-1}^2+1)(L_k^2+1)}{(L_{k-1}L_k+1)}\geq 2L_k. \tag{3}$$

Note that the result is true for $k=1$; not true for $k=2$, but for all $k\geq 3$ the above relation is true. This would mean the problem, as originally stated,

$$\frac{(L_1^2+1)(L_2^2+1)}{L_1L_2+1}+\frac{(L_2^2+1)(L_3^2+1)}{L_2L_3+1}+\cdots+\frac{(L_{n-1}^2+1)(L_n^2+1)}{L_{n-1}L_n+1}+\frac{(L_n^2+1)(L_1^2+1)}{L_nL_1+1}\geq 2L_{n+2}-6,$$

is certainly true, where n is a positive integer ≥ 2.

Next, we will note that the LHS of (3) may be written as

$$(L_{k-1}L_k + 1) + \frac{L_{k-2}^2}{L_{k-1}L_k + 1}. \tag{4}$$

This follows from dividing the bottom into the top and then we prove

$$L_{k-1}^2 + L_k^2 - 2L_{k-1}L_k = L_{k-2}^2. \tag{5}$$

This easily follows by writing L_{k-2} as $L_k - L_{k-1}$; then squaring and comparing terms.

Since L_{k-1} exceeds 2 for all $k \geq 3$, this clearly establishes (3) and we are done.

References

[1] V. E. Hoggatt, Jr., *Fibonacci and Lucas Numbers*, The Fibonacci Association, Santa Clara, CA, 1979.

Also solved by Paul S. Bruckman, Charles K. Cook, Dmitry Freishman, Ángel Plaza, and the proposer.

We would like to acknowledge the solution of B-1115 by Anastasios Kotronis.