ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
RUSS EULER AND JAWAD SADEK

Please submit all new problem proposals and their solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468, or by email at reuler@nwmissouri.edu. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by May 15, 2014. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2,\ \beta = (1-\sqrt{5})/2,\ F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1136 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Prove that

$$\sum_{k=1}^{n} (F_k F_{k+1})^3 = \left(\sum_{k=1}^{n} F_k^2 F_{k+1} \right)^2.$$

B-1137 **Proposed by D. M. Bătineţu–Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.**

Prove that

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\frac{L_m^k+L_{m+1}^k}{L_{m+2}^k}=\frac{L_m^{2n}+L_{m+1}^{2n}}{L_{m+2}^{2n}} \text{ for any positive integer } n; \tag{1}$$

and

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\frac{F_m^k+F_{m+1}^k}{F_{m+2}^k}=\frac{F_m^{2n}+F_{m+1}^{2n}}{F_{m+2}^{2n}} \text{ for any positive integer } n. \tag{2}$$

B-1138 **Proposed by D. M. Bătineţu–Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.**

Prove that if $m>0$ and $p>0$, then

$$\left(\sum_{k=1}^{n}\frac{F_k^{m+1}}{L_k^m}\right)\left(\sum_{k=1}^{n}\frac{F_k^{p+1}}{L_k^p}\right)\geq\frac{(F_{n+2}-1)^{m+p+2}}{(L_n-3)^{m+p}},$$

for any positive integer n.

B-1139 **Proposed by D. M. Bătineţu–Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.**

Prove that

$$\sum_{k=1}^{n}(1+F_k^2+L_k^2)^2>4(F_nF_{n+1}+L_nL_{n+1})-8,$$

for any positive integer n.

B-1140 **Proposed by José Luis Díaz-Barrero, BARCELONA TECH, Barcelona, Spain.**

Let $n\geq 2$ be a positive integer. Show that

$$\frac{1}{2}\left(\frac{F_n^3}{F_{n+1}(F_{n+1}-F_n)}+\frac{F_{n+1}^3}{F_n(F_n-F_{n+1})}+\frac{F_{n+2}^3}{F_{n+1}(F_{n+2}-F_{n+1})}\right)$$

is an integer and determine its value.

SOLUTIONS

In recognition of his invaluable contributions to the Fibonacci Quarterly, we are dedicating this issue to the late Paul S. Bruckman. Paul had solved all of the problems appearing in this section during our tenure as co-editors, and we are featuring his solutions to all the problems in this issue. He will be missed.

Another Division by 5

B-1116 Proposed by M. N. Deshpande, Nagpur, India. (Vol. 50.4, November 2012)

Let n be a nonnegative integer and let T_n be the nth triangular number. Prove that each of the following is divisible by 5:

$$nL_{n+1} + 2F_n \tag{1}$$

$$T_nL_{n+1} + (n+1)F_n \tag{2}$$

Solution by Paul S. Bruckman, Nanaimo, BC, Canada.

The sequence $\{L_n \pmod 5\}_{n=1}^{\infty}$ is periodic with period 4; the (repeating) period is $\{1, 3, 4, 2\}$. This is the same as the sequence $\{L_{n+1} \pmod 5\}_{n=0}^{\infty}$. The sequence $\{nL_{n+1} \pmod 5\}_{n=0}^{\infty}$ is readily seen to be periodic with period 20. We find that its repeating period is as follows: $\{0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1, 0, 1, 1, 2, 3\}$. Also, the sequence $\{2F_n \pmod 5\}_{n=0}^{\infty}$ is periodic with period 20. Its repeating period is seen to be as follows: $\{0, 2, 2, 4, 1, 0, 1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2\}$. It then readily follows that the sequence $\{(nL_{n+1} + 2F_n) \pmod 5\}_{n=0}^{\infty}$ is periodic with a period of 1, and its repeating period is $\{0\}$. That is, the first given sequence is divisible by 5 for all $n \geq 0$.

If $u_n = nL_{n+1} + 2F_n$ and $v_n = T_nL_{n+1} + (n+1)F_n$, we have shown that $5|u_n$ for all $n \geq 0$. Since $T_n = n(n+1)/2$, we see that $v_n = \frac{(n+1)}{2}u_n$. If n is odd, this immediately implies that $5|v_n$ for all odd $n \geq 1$. However, v_n is clearly an integer. It must therefore be true that $5|v_n$ for all even $n \geq 0$, which proves that $5|v_n$ for all $n \geq 0$.

Also solved by Scott H. Brown, Michael R. Bacon and Charles C. Cook (jointly), Edwardo H. M. Brietzke, Kenneth B. Davenport, Dmitry Fleischman, Amos E. Gera, Ralph P. Grinaldi, Russell Jay Hendel, Robinson Higuita, Zbigniew Jacubczyck, Parviz Khalili, Harris Kwong, Kathleen E. Lewis, Zachary McCaslin, Ángel Plaza and Sergio Falcón (jointly), David Stone and John Hawkins (jointly), Rattanapol Wasutharat, and the proposer.

Two Sums with Square Roots

B-1117 Proposed by D. M. Bătineţu–Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania. (Vol. 50.4, November 2012)

Prove that:

$$\sum_{k=1}^{n}\left(\sqrt{F_k^4-F_k^2+1}+\frac{F_k^2-1}{F_k^4+1}\right)<F_nF_{n+1} \tag{1}$$

$$\sum_{k=1}^{n}\left(\sqrt{L_k^4-L_k^2+1}+\frac{L_k^2-1}{L_k^4+1}\right)<L_nL_{n+1}-2 \tag{2}$$

for any positive integer n.

Solution by Paul S. Bruckman, Nanaimo, BC, Canada.

Let

$$S_n=\sum_{k=1}^{n}\left(\sqrt{F_k^4-F_k^2+1}+\frac{F_k^2-1}{F_k^4+1}\right), T_n=\sum_{k=1}^{n}\left(\sqrt{L_k^4-L_k^2+1}+\frac{L_k^2-1}{L_k^4+1}\right), n=1,2,\ldots. \tag{1}$$

We also use the following known identities:

$$\sum_{k=1}^{n}F_k^2=F_nF_{n+1};\sum_{k=1}^{n}L_k^2=L_nL_{n+1}-2. \tag{2}$$

It suffices to prove the following, valid for all $x\geq 1$:

$$\sqrt{x^2-x+1}+\frac{x-1}{x^2+1}\leq x. \tag{3}$$

For if (3) is true, then setting $x=F_k^2$ or $x=L_k^2$ in (3) and using (2) implies $S_n\leq F_nF_{n+1}$, $T_n\leq L_nL_{n+1}-2$. This is a slightly weaker result than the desired original, since we find that equality holds for $n=1$ (i.e. $x=1$).

Proof of (3). Given $x\geq 1$, let

$$G(x)=x-\sqrt{x^2-x+1}-\frac{x-1}{x^2+1}=\frac{x^3+1}{x+1}-\sqrt{x^2-x+1}=\frac{x^3+1}{x+1}-\sqrt{\frac{x^3+1}{x+1}}$$

which is manifestly ≥ 0 for all $x\geq 1$. This proves (3). □

Also solved by Charles C. Cook, Kenneth B. Davenport, Dmitry Fleischman, Amos E. Gera, Russell Jay Hendel, Robinson Higuita, Zbigniew Jacubczyck, Parviz Khalili, Harris Kwong, Kathleen E. Lewis, Ángel Plaza, Marielle Silvio and Kasey Zemba (jointly), and the proposer.

Quadratic Identities

B-1118 Proposed by Gordon Clarke, Brisbane, Australia. (Vol. 50.4, November 2012)

If n is a nonnegative integer, prove that:

$$\left(F_n^2+F_{n+1}^2\right)\left(F_{n+2}^2+F_{n+3}^2\right)=F_{2n+3}^2+1 \tag{1}$$

$$\left(F_n^2+F_{n+2}^2\right)\left(F_{n+4}^2+F_{n+6}^2\right)=F_{2n+6}^2+\left(2F_{n+3}^2\pm 5\right)^2. \tag{2}$$

Solution by Paul S. Bruckman, Nanaimo, BC, Canada.

Part (1). We use the following known identity:

$$F_n^2 + F_{n+1}^2 = F_{2n+1}.$$

Then also, $F_{n+1}^2 + F_{n+3}^2 = F_{2n+5}$. Therefore, the left side of (1) equals $F_{2n+1}F_{2n+5}$. We now use the following known identity: $F_mF_{m+4} - F_{m+2}^2 = (-1)^{m-1}$. Therefore, setting $m = 2n+1$ in the left side of (1) yields $F_{2n+1}F_{2n+5} = F_{2n+3}^2 + 1$.
Part (2). We use the following known identities:

$$F_n^2 + F_{n+2}^2 = 3F_{n+1}^2 - 2(-1)^n \quad \text{and} \quad F_m^2 + F_{m+4}^2 = 7F_{m+2}^2 + 2(-1)^m.$$

Then also $F_{n+4}^2 + F_{n+6}^2 = 3F_{n+5}^2 - 2(-1)^n$. The left side of (2) then becomes

$$\{3F_{n+1}^2 - 2(-1)^n\}\{3F_{n+5}^2 - 2(-1)^n\} = 9(F_{n+1}F_{n+5})^2 - 6(-1)^n\{F_{n+1}^2F_{n+5}^2\} + 4;$$

from Part (1), with $m = n+1$, $F_{n+1}F_{n+5} = F_{n+3}^2 + (-1)^n$. Also, with $m = n+1$, we obtain $F_{n+1}^2 + F_{n+5}^2 = 7F_{n+3}^2 - 2(-1)^n$. Then the left side of (2) equals

$$9\{F_{n+3}^2 + (-1)^n\}^2 - 6(-1)^n\{F_{n+1}^2 + F_{n+5}^2\} + 4.$$

The left side of (2) then becomes

$$\begin{aligned}
&9\{F_{n+3}^2 + (-1)^n\}^2 - 6(-1)^n\{7F_{n+3}^2 - 2(-1)^n\} + 4\\
&\quad = 9F_{n+3}^4 + (18-42)(-1)^nF_{n+3}^2 + (9+12+4)\\
&\quad = 9F_{n+3}^4 - 24(-1)^nF_{n+3}^2 + 25\\
&\quad = 5F_{n+3}^4 - 4(-1)^nF_{n+3}^2 + \{2F_{n+3}^2 - 5(-1)^n\}^2\\
&\quad = F_{n+3}^2\{5F_{n+3}^2 - 4(-1)^n\} + \{2F_{n+3}^2 - 5(-1)^n\}^2\\
&\quad = F_{n+3}^2L_{n+3}^2 + \{2F_{n+3}^2 - 5(-1)^n\}^2\\
&\quad = F_{2n+6}^2 + \{2F_{n+3}^2 - 5(-1)^n\}^2.
\end{aligned}$$

George A. Heisert used a nontraditional approach to solve the problem. The technique (called dual polynomial method) is explained in detail in his article "A different method for Deriving Fibonacci Power Identities," JP Journal of Algebra, Number Theory and Applications, **24** (2012), 1–26.

Also solved by Charles C. Cook, Kenneth B. Davenport, Dmitry G. Fleishman, Amos E. Gera, George A. Heisert, Russell Jay Hendel, Zbigniew Jakubczyk, Harris Kwong, Carl Libis, Ángel Plaza and Sergio Falcón (jointly), and the proposer.

A Trig and Fibonacci Amalgam

B-1119 Proposed by D. M. Bătineţu–Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.
(Vol. 50.4, November 2012)

Prove that

$$\sum_{k=1}^{n} \frac{\tan^2 \frac{x}{2^k}}{2^k F_k^2} \geq \frac{\left(\cot \frac{x}{2^n} - 2^{n+1} \cot 2x\right)^2}{2^{2n} F_n F_{n+1}}, \text{ for any } x \in \left(0, \frac{\pi}{4}\right).$$

Solution by Paul S. Bruckman, Nanaimo, BC, Canada.

In the following manipulations, all trigonometric functions are well-defined and positive in the open interval $x \in (0, \pi/4)$. We first observe that the proposed inequality, as presently stated, is false. To see this, we let $n = 1$, $x = \pi/6$.

The left member of the proposed inequality is $\frac{1}{2}\tan^2\left(\frac{\pi}{12}\right) = 3.5 - 2\sqrt{3} \approx 0.036$; the right member is $\{\cot(\pi/12) - 4\cot(\pi/3)\}^2/4 = (13 - 4\sqrt{3})/12 \approx 0.506$. Thus, the proposed inequality is clearly false.

We believe that the proposer intended to show the following inequality:

$$\sum_{k=1}^{n} \frac{\tan^2 \frac{x}{2^k}}{2^{2k} F_k^2} \geq \left\{\cot\left(\frac{x}{2^n}\right) - 2^n \cot(x)\right\}^2 / 2^{2n} F_n F_{n+1}, n = 1, 2, \ldots, \text{ for all } x \in (0, \pi/4). \quad (1)$$

We first make the following definitions for $k = 1, 2, \ldots, n$: $u_k = \frac{x}{2^k}$; $a_k = \frac{\tan u_k}{2^k F_k}$, $b_k = F_k$; therefore, $a_k b_k = \frac{\tan u_k}{2^k}$. Let

$$S = S(x, n) = \sum_{k=1}^{n} \frac{\tan^2 u_k}{2^{2k} F_k^2} = \sum_{k=1}^{n} a_k^2.$$

We also recall the following well-known identity:

$$\sum_{k=1}^{n} b_k^2 = \sum_{k=1}^{n} F_k^2 = F_n F_{n+1}.$$

We now invoke the Cauchy-Schwarz inequality:

$$\left\{\sum_{k=1}^{n} a_k b_k\right\}^2 \leq \left\{\sum_{k=1}^{n} a_k^2\right\}\left\{\sum_{k=1}^{n} b_k^2\right\}.$$

Therefore, $\left\{\sum_{k=1}^{n} \frac{\tan u_k}{2^k}\right\}^2 \leq F_n F_{n+1} S$ or

$$S \geq \left\{\sum_{k=1}^{n} \frac{\tan u_k}{2^k}\right\}^2 / F_n F_{n+1}. \quad (2)$$

We now employ the following trigonometric identity, valid for $z \in (0, \pi/2)$

$$\cot z - 2\cot 2z = \tan z. \quad (3)$$

We may easily verify (3) from the definitions of the indicated functions.

Setting $z = u_k$, we then have $\tan u_k = \cot u_k - 2\cot 2u_k = \cot\left(\frac{x}{2^k}\right) - 2\cot\left(\frac{x}{2^{k-1}}\right)$; then

$$\frac{\tan u_k}{2^k} = \frac{\cot(x/2^k)}{2^k} - \frac{\cot(x/2^{k-1})}{2^{k-1}}.$$

From this, it follows by telescoping that

$$\sum_{k=1}^{n} \frac{\tan u_k}{2^k} = \frac{\cot u_n}{2^n} - \cot x. \quad (4)$$

Proof of (1). By (2) and (4),

$$S \geq \left\{ \frac{\cot\left(\frac{x}{2^n}\right)}{2^n} - \cot(x) \right\}^2 / F_n F_{n+1}, \quad \text{or} \quad S \geq \left\{ \cot\left(\frac{x}{2^n}\right) - 2^n \cot(x) \right\}^2 / 2^{2n} F_n F_{n+1}. \tag{5}$$

Equality occurs if and only if $n = 1$; in this case both sides are equal to

$$\left(\frac{\tan(x/2)}{2}\right)^2 = \left(\frac{\cot(x/2) - 2\cot(x)}{2}\right)^2.$$

For $n = 1$ and $x = \pi/6$, each side equals $1.7 - \sqrt{3} \approx .0179492$.

Higuita took a different route and corrected the proposed inequality by starting the sum at $k = 0$.

Also solved by Kenneth B. Davenport, Dmitry Fleischman, Robinson Higuita, and the proposer.

Periodic Sequences

B-1120 Proposed by the Problem Editor.
(Vol. 50.4, November 2012)

Prove or disprove: $F_n \equiv 2n3^n \pmod 5$ for all nonnegative integers n.

Solution by Paul S. Bruckman, Nanaimo, BC, Canada.

The sequence $\{F_n \pmod 5\}_{n=0}^{\infty}$ is periodic with period 20. Its repeating period is found to be as follows: $\{0, 1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1\}$. On the other hand, the sequence $\{2n \pmod 5\}_{n=0}^{\infty}$ is periodic with period 5; its repeating period is $\{0, 2, 4, 1, 3\}$. Also the sequence $\{3^n \pmod 5\}_{n=0}^{\infty}$ is periodic with period 4; its repeating period is $\{1, 3, 4, 2\}$. Therefore, the sequence $\{2n3^n \pmod 5\}_{n=0}^{\infty}$ is periodic with period 20 equal to GCM(4,5). We find that its repeating period is the same as that of $\{F_n \pmod 5\}_{n=0}^{\infty}$. Therefore, $F_n \equiv 2n3^n \pmod 5$ for all $n \geq 0$. The conjecture is true.

Also solved by Edwardo H. M. Brietzke, Michael J. Buckmarter, Charles C. Cook, Kenneth B. Davenport, Dmitry Fleischman, Amos E. Gera, Ralph P. Grimaldi, Russell J. Hendel, Robinson Higuita, Harris Kwong, Kathleen E. Lewis, Carl Libis, Marielle Silvio and Kasey Zemba (jointly), David Stone and John Hawkins (jointly), Matt Zinkle, and the proposer.

Solution to Problem 1114 was published in the August, 2013 issue. We list here the name of additional solvers: Brian Beasley, Paul S. Bruckman, Russell Jay Hendel, Gurdial Aroroa and Sindhu Unnithan (jointly).

We would like to belatedly acknowledge Charles Cook for solving problem B-1112.