ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
RUSS EULER AND JAWAD SADEK

Please submit all new problem proposals and their solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468, or by email at reuler@nwmissouri.edu. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by May 15, 2016. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1176 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Find a closed form expression for the sum

$$\sum_{k=0}^{n} L_{3^k}^3 + 2\sum_{k=0}^{n} L_{3^k}.$$

B-1177 Proposed by Ángel Plaza and Sergio Falcón, Universidad de Las Palmas de Gran Canaria, Spain.

For any integer $p \geq 0$, prove each of the following:

(i) $\lim_{n\to\infty} \dfrac{F_n^p + F_{n+3}^p}{F_{n+2}^p} = \dfrac{L_{2p}+L_p}{2} - \dfrac{F_{2p}-F_p}{2}\sqrt{5}$

(ii) $\lim_{n\to\infty} \dfrac{L_n^p + L_{n+3}^p}{L_{n+2}^p} = \dfrac{L_{2p}+L_p}{2} - \dfrac{F_{2p}-F_p}{2}\sqrt{5}.$

B-1178 Proposed by D. M. Bătineţu–Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.

Prove that

$$\sqrt{F_1^4 - F_1^2F_2^2 + F_2^4} + \sqrt{F_2^4 - F_2^2F_3^2 + F_3^4} + \cdots$$
$$+ \sqrt{F_{n-1}^4 - F_{n-1}^2F_n^2 + F_n^4} + \sqrt{F_n^4 - F_n^2 + 1} > F_nF_{n+1}.$$

B-1179 Proposed by D. M. Bătineţu–Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.

Let $\{x_n\}_{n\geq 1}$ be a sequence of real numbers. Prove that

$$2 \cdot \left(\sum_{k=1}^{n} F_k \cdot \sin x_k\right) \cdot \left(\sum_{k=1}^{n} F_k \cdot \cos x_k\right) \leq n \cdot F_n \cdot F_{n+1}.$$

B-1180 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

If $i = \sqrt{-1}$, find

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n}+i}.$$

SOLUTIONS

A Trig Trick

B-1156 Proposed by Hideyuki Ohtsuka, Saitama, Japan. (Vol. 52.4, November 2014)

Prove that

$$\sum_{n=0}^{\infty} \tan^{-1}\left(\frac{\sqrt{5}}{L_{4n+2}}\right) = \frac{\pi}{4}.$$

Solution by Albert Stadler, 8704 Herrliberg, Switzerland.

We claim that

$$\sum_{j=0}^{k} \arctan\left(\frac{\sqrt{5}}{L_{4j+2}}\right) = \arctan\left(\frac{\sqrt{5}F_{2k+2}}{L_{2k+2}}\right), \tag{1}$$

for $k = 0, 1, 2, \ldots$, from which the assertion follows immediately, since $\lim\limits_{k\to\infty} \dfrac{\sqrt{5}F_{2k+2}}{L_{2k+2}} = 1$, and $\arctan(1) = \frac{\pi}{4}$.

We proceed by induction. Equation (1) is true for $k = 0$. Suppose it is true for a specific k. Then

$$\begin{aligned}
\sum_{j=0}^{k+1} \arctan\left(\frac{\sqrt{5}}{L_{4j+2}}\right) &= \arctan\left(\frac{\sqrt{5}}{L_{4k+6}}\right) + \sum_{j=0}^{k} \arctan\left(\frac{\sqrt{5}}{L_{4j+2}}\right)\\
&= \arctan\left(\frac{\sqrt{5}}{L_{4k+6}}\right) + \arctan\left(\frac{\sqrt{5}F_{2k+2}}{L_{2k+2}}\right)\\
&= \arctan\left(\frac{\frac{\sqrt{5}}{L_{4k+6}} + \frac{\sqrt{5}F_{2k+2}}{L_{2k+2}}}{1 - \frac{\sqrt{5}}{L_{4k+6}} \cdot \frac{\sqrt{5}F_{2k+2}}{L_{2k+2}}}\right)\\
&= \arctan\left(\sqrt{5}\frac{L_{2k+2} + F_{2k+2}L_{4k+6}}{L_{4k+6}L_{2k+2} - 5F_{2k+2}}\right)\\
&= \arctan\left(\frac{\sqrt{5}F_{2k+4}}{L_{2k+4}}\right),
\end{aligned}$$

if we are able to prove that

$$\frac{L_{2k+2} + F_{2k+2}L_{4k+6}}{L_{4k+6}L_{2k+2} - 5F_{2k+2}} = \frac{F_{2k+4}}{L_{2k+4}}$$

or equivalently

$$(L_{2k+2} + F_{2k+2}L_{4k+6})L_{2k+4} = (L_{4k+6}L_{2k+2} - 5F_{2k+2})F_{2k+4}. \tag{2}$$

But equation (2) follows easily by using

$$F_{n+2} = F_{n+1} + F_n, F_0 = 0, F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, L_0 = 2, L_1 = 1;$$
$$\alpha = (1+\sqrt{5})/2, \beta = (1-\sqrt{5})/2, F_n = (\alpha^n - \beta^n)/\sqrt{5}, \quad \text{and} \quad L_n = \alpha^n + \beta^n;$$

and multiplying out.

Also solved by Brian Bradie, Kenneth B. Davenport, Dmitry Fleischman, Russell Jay Hendel, Ángel Plaza, and the proposer.

A Cyclic Sum

B-1157 Proposed by Hideyuki Ohtsuka, Saitama, Japan. (Vol. 52.4, November 2014)

For positive integers a, b, c prove that

$$\frac{F_{2a}}{F_{b+c}} + \frac{F_{2b}}{F_{c+a}} + \frac{F_{2c}}{F_{a+b}} \geq 3.$$

Solution by Wei-Kai Lai, University of South Carolina Salkehatchie.

Since $F_{2n} = F_nL_n$, and $2F_{m+n} = F_mL_n + F_nL_m$ (Ferns, 1967), the proposed inequality is equivalent to

$$\frac{F_aL_a}{F_bL_c + F_cL_b} + \frac{F_bL_b}{F_cL_a + F_aL_c} + \frac{F_cL_c}{F_aL_b + F_bL_a} \geq \frac{3}{2}.$$

Without loss of generality, we may assume that $a \geq b \geq c$. Accordingly, $F_a \geq F_b \geq F_c$ and $L_a \geq L_b \geq L_c$. Based on the rearrangement inequality,

$$F_bL_b + F_cL_c \geq F_bL_c + F_cL_b, \quad F_cL_c + F_aL_a \geq F_cL_a + F_aL_c, \quad F_aL_a + F_bL_b \geq F_aL_b + F_bL_a.$$

Therefore,

$$\frac{F_aL_a}{F_bL_c + F_cL_b} + \frac{F_bL_b}{F_cL_a + F_aL_c} + \frac{F_cL_c}{F_aL_b + F_bL_a} \geq \frac{F_aL_a}{F_bL_b + F_cL_c} + \frac{F_bL_b}{F_cL_c + F_aL_a} + \frac{F_cL_c}{F_aL_a + F_bL_b}.$$

For the right side of the above inequality, we already know that

$$\frac{F_aL_a}{F_bL_b + F_cL_c} + \frac{F_bL_b}{F_cL_c + F_aL_a} + \frac{F_cL_c}{F_aL_a + F_bL_b} \geq \frac{3}{2},$$

which is the Nesbitt's inequality. The proposed inequality is then proved with equality when $a = b = c$.

Also solved by Dmitry Fleischman, Russell Jay Hendel, Ángel Plaza, and the proposer.

Inequality of Powers

B-1158 Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade School, Buzău, Romania.
(Vol. 52.4, November 2014)

If $a, b, m > 0$, then prove that

$$\sum_{k=1}^{n} \frac{1}{\left(a \cdot F_k^2 + b \cdot \sqrt[n]{\prod_{k=1}^{n} F_k^2}\right)^m} \geq \frac{n^{m+1}}{(a+b)^m F_n^m F_{n+1}^m} \tag{1}$$

and

$$\sum_{k=1}^{n} \frac{1}{\left(a \cdot L_k^2 + b \cdot \sqrt[n]{\prod_{k=1}^{n} L_k^2}\right)^m} \geq \frac{n^{m+1}}{(a+b)^m (L_nL_{n+1} - 2)^m} \tag{2}$$

for any positive integer n.

Solution by Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania.

Applying Radon's inequality we get:

$$\sum_{k=1}^{n} \frac{1}{(aF_k^2 + b\sqrt[n]{\prod_{k=1}^{n} F_k^2})^m} \geq \frac{(1+1+1+\cdots 1)^{m+1}}{(\sum_{k=1}^{n} aF_k^2 + nb\sqrt[n]{\prod_{k=1}^{n} F_k^2})^m}$$
$$= \frac{n^{m+1}}{(a\sum_{k=1}^{n} F_k^2 + b\sum_{k=1}^{n} F_k^2)^m}$$
$$= \frac{n^{m+1}}{(a+b)^m F_n^m F_{n+1}^m},$$

where $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$.

Similarly,

$$\sum_{k=1}^{n} \frac{1}{(aL_k^2 + b\sqrt[n]{\prod_{k=1}^{n} L_k^2})^m} \geq \frac{(1+1+1+\cdots 1)^{m+1}}{(\sum_{k=1}^{n} aL_k^2 + nb\sqrt[n]{\prod_{k=1}^{n} L_k^2})^m}$$
$$= \frac{n^{m+1}}{(a\sum_{k=1}^{n} L_k^2 + b\sum_{k=1}^{n} L_k^2)^m}$$
$$= \frac{n^{m+1}}{(a+b)^m (L_n L_{n+1} - 2)^m},$$

where $\sum_{k=1}^{n} L_k^2 = L_n L_{n+1} - 2$.

Also solved by Brian Bradie, Dmitry Fleischman, Hideyuki Ohtsuka, Ángel Plaza, and the proposer.

Inequality With Sum of Inverses

B-1159 Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade School, Buzău, Romania.
(Vol. 52.4, November 2014)

If $x, y, z > 0$, then prove that

$$\frac{x}{xF_n + yF_{n+1} + zF_{n+2}} + \frac{y}{yF_n + zF_{n+1} + xF_{n+2}} + \frac{z}{zF_n + xF_{n+1} + yF_{n+2}} \geq \frac{3}{2F_{n+2}} \quad (1)$$

and

$$\frac{x}{xL_n + yL_{n+1} + zL_{n+2}} + \frac{y}{yL_n + zL_{n+1} + xL_{n+2}} + \frac{z}{zL_n + xL_{n+1} + yL_{n+2}} \geq \frac{3}{2L_{n+2}} \quad (2)$$

for any positive integer n.

Solution by Adnan Ali, Student, A.E.C.S-4, Mumbai, India.

From the Cauchy-Schwartz Inequality,

$$\frac{x^2}{x^2F_n + xyF_{n+1} + zxF_{n+2}} + \frac{y^2}{y^2F_n + yzF_{n+1} + xyF_{n+2}} + \frac{z^2}{z^2F_n + zxF_{n+1} + yzF_{n+2}}$$
$$\geq \frac{(x+y+z)^2}{F_n(x^2+y^2+z^2) + F_{n+1}(xy+yz+zx) + F_{n+2}(xy+yz+zx)}.$$

So, if suffices to prove that

$$\frac{(x+y+z)^2}{F_n(x^2+y^2+z^2)+F_{n+1}(xy+yz+zx)+F_{n+2}(xy+yz+zx)} \geq \frac{3}{2F_{n+2}} = \frac{3}{2F_n+2F_{n+1}}.$$

This is equivalent to

$$\begin{aligned}2(x+y+z)^2F_n+2(x+y+z)^2F_{n+1} &\geq 3(x^2+y^2+z^2)F_n+3(xy+yz+zx)F_{n+1}\\ &\quad +3F_{n+2}(xy+yz+zx) \geq 3(x^2+y^2+z^2)F_n+3(xy+yz+zx)F_{n+1}\\ &\quad +3F_n(xy+yz+zx)+3F_{n+1}(xy+yz+zx).\end{aligned}$$

By rearranging the terms, the inequality is equivalent to

$$\begin{aligned}&F_n(xy+yz+zx-x^2-y^2-z^2)+2F_{n+1}(x^2+y^2+z^2-xy-yz-zx)\\ &\quad=(2F_{n+1}-F_n)(x^2+y^2+z^2-xy-yz-zx)\geq 0\end{aligned}$$

which is true since $x^2+y^2+z^2-xy-yz-zx=\frac{1}{2}((x-y)^2+(y-z)^2+(z-x)^2)\geq 0$.

Equation (2) can be obtained by replacing F_n by L_n in the above proof.

Also solved by Brian Bradie, Dmitry Fleischman, Russell Jay Hendel, Ángel Plaza, Nicuşor Zlota, and the proposer.

More Nuances Than the Limits in B-1151

B-1160 Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade School, Buzău, Romania.
(Vol. 52.4, November 2014)

If $e_n=(1+\frac{1}{n})^n$, then $\lim_{n\to\infty}e_n=e$. Compute each of the following:

(1) $\lim\limits_{n\to\infty}\left(e\cdot\sqrt[n+1]{(n+1)!F_{n+1}}-e_n\cdot\sqrt[n]{n!F_n}\right)$

(2) $\lim\limits_{n\to\infty}\left(e_{n+1}\cdot\sqrt[n+1]{(n+1)!F_{n+1}}-e_n\cdot\sqrt[n]{n!F_n}\right)$

(3) $\lim\limits_{n\to\infty}\left(e\cdot\sqrt[n+1]{(n+1)!L_{n+1}}-e_n\cdot\sqrt[n]{n!L_n}\right)$

(4) $\lim\limits_{n\to\infty}\left(e_{n+1}\cdot\sqrt[n+1]{(n+1)!L_{n+1}}-e_n\cdot\sqrt[n]{n!L_n}\right)$.

Solution by Kenneth B. Davenport, Dallas, PA.

To solve this problem, we utilize the following relations:

$$\lim_{n\to\infty}\frac{(n!)^{1/n}}{n}=\frac{1}{e} \tag{a}$$

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= e^{1-1/2n+0(1/n^2)}\\ &= e\left[1-\frac{1}{2n}+0\left(\frac{1}{n^2}\right)\right].\end{aligned} \tag{b}$$

For (a), see [1, p. 524]; and for (b), see P. Bruckman's solution in [2, No. 908, p. 672].

For (1), we may re-express the limit, noting that for large n

$$F_n \approx \frac{a^n}{\sqrt{5}}, \tag{c}$$

as

$$\alpha \cdot \lim_{n\to\infty} \left(e\sqrt[n+1]{(n+1)!} - e\left[1 - \frac{1}{2n} + 0\left(\frac{1}{n^2}\right)\right]\sqrt[n]{n!}\right). \tag{d}$$

Since

$$\lim_{n\to\infty} \frac{1}{(\sqrt{5})^{1/n}} = 1.$$

Using equations (a), (d) becomes

$$\alpha \cdot \lim_{n\to\infty} \left[(n+1) - \left(n - \frac{1}{2} + 0\left(\frac{1}{n}\right)\right)\right].$$

The desired limit is thus $\frac{3}{2}$.

Similarly, in part (2) we may re-express the limit as follows:

$$\alpha \cdot \lim_{n\to\infty} \left[e\left(1 - \frac{1}{2(n+1)} + 0\left(\frac{1}{n^2}\right)\right)\sqrt[n+1]{(n+1)!} - e\left(1 - \frac{1}{2n} + 0\left(\frac{1}{n^2}\right)\right)\sqrt[n]{n!}\right].$$

This then becomes

$$\alpha \cdot \lim_{n\to\infty} \left[(n+1) - \frac{1}{2} + 0\left(\frac{1}{n}\right) - \left(n - \frac{1}{2} + 0\left(\frac{1}{n}\right)\right)\right].$$

Thus, our desired limit is simply α.

Switching L_n for F_n; and L_{n+1} for F_{n+1} in parts (3) and (4) does not materially alter the computation of the respective limits. We will still get $3/2$ for part (3) and α as the result for part (4).

References

[1] *The Advanced Calculus Problem Solver*, Research and Education Association, 1981.
[2] *Pi Mu Epsilon Journal*, Vol. 10, No. 8, Spring 1998.

Also solved by Dmitry Fleischman, Ángel Plaza, and the proposer.

We would like to belatedly acknowledge, Dmitry Fleischman for solving B-1153.