ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Florian Luca

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, IMATE, UNAM, AP. POSTAL 61-3 (XANGARI), CP 58 089, MORELIA, MICHOACAN, MEXICO, or by e-mail at fluca@matmor.unam.mx as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-635 Proposed by Jayantibhai M. Patel, Ahmedabad, India

For any positive integer n, prove that

$$\begin{vmatrix} F_n & F_{n+1} & F_{n+2} \\ F_{n+2} & F_n & F_{n+1} \\ F_{n+1} & F_{n+2} & F_n \end{vmatrix} = 2(F_n^3 + F_{n+1}^3),$$

and that the same holds with the Fibonacci numbers replaced by the corresponding Lucas numbers.

H-636 Proposed by Charles K. Cook, Sumter, SC

Evaluate the determinant of the matrix

$$\begin{vmatrix} F_n^2 + L_n^2 - P_n^2 - R_n^2 & 2(L_nP_n - F_nR_n) & 2(F_nP_n + L_nR_n) \\ 2(F_nR_n + L_nP_n) & F_n^2 - L_n^2 + P_n^2 - R_n^2 & 2(P_nR_n - F_nL_n) \\ 2(L_nR_n - F_nP_n) & 2(F_nL_n + P_nR_n) & F_n^2 - L_n^2 - P_n^2 + R_n^2 \end{vmatrix}$$

where F_n, L_n, P_n and R_n are the Fibonacci, Lucas, Pell and Pell-Lucas numbers, respectively.

H-637 Proposed by Ovidiu Furdui, Kalamazoo, MI

Prove that

$$\sqrt{\frac{1}{2} + \frac{F_{2n}}{2\sqrt{F_{2n}^2 + 1}}}, \quad \sqrt{\frac{1}{2} + \frac{1}{2\sqrt{F_{2n+1}^2 + 1}}} \text{ and } \sqrt{\frac{1}{2} + \frac{1}{2\sqrt{F_{2n+2}^2 + 1}}}$$

are the sides of a triangle whose circumradius is 1/2 for all $n \geq 0$.

H-638 Proposed by José Luis Díaz-Barrero, Barcelona, Spain

Let n be a positive integer. Prove that

$$4+2\sum_{k=1}^{n}\frac{F_{k+1}}{\log\bigl(1+F_{k+1}/F_k\bigr)}<F_{n+1}+3F_{n+2}.$$

SOLUTIONS

An identity for Lucas polynomials

H-621 Proposed by Mario Catalani, Torino, Italy (Vol. 43, no. 1, February 2005)

Let $L_n(x,y)$ be the bivariate Lucas polynomials, defined by $L_n(x,y) = xL_{n-1}(x,y) + yL_{n-2}(x,y),\ L_0(x,y)=2,\ L_1(x,y)=x$. Assume $x^2+4y\neq 0$. Prove the following identity

$$\sum_{k=0}^{n}\binom{n+k}{k}(-y)^k x^{-(k+1)}L_{n+1-k}(x,y)=x^n.$$

Solution by the proposer

Consider the polynomials $L_n(1,\,y)$. The roots of the characteristic equation, $\alpha\equiv\alpha(1,\,y)$ and $\beta\equiv\beta(1,\,y)$, are such that $\alpha+\beta=1$, $\alpha\beta=-y$.
Identity 1.78 in [2] says that

$$\sum_{k=0}^{n}\binom{n+k}{k}\left[(1-x)^{n+1}x^k+x^{n+1}(1-x)^k\right]=1.$$

In the left hand side, write $x=\alpha$ to get

$$\sum_{k=0}^{n}\binom{n+k}{k}(\beta^{n+1}\alpha^k+\alpha^{n+1}\beta^k)=\sum_{k=0}^{n}\binom{n+k}{k}(-y)^kL_{n+1-k}(1,\,y). \tag{1}$$

In [1], it is proved that

$$L_n(x,\,-y)=(\mathrm{i}\gamma)^{-n}L_n(\mathrm{i}\gamma x,\,\gamma^2 y),$$

where $\mathrm{i}^2=-1$ and γ is a complex number. Take $\gamma=\frac{1}{\mathrm{i}x}$. Then

$$L_n(x,\,-y)=x^nL_n\left(1,\,-\frac{y}{x^2}\right),$$

that is

$$L_n(x,\, y) = x^n L_n\left(1,\, \frac{y}{x^2}\right). \tag{2}$$

In the right hand side of formula (1) replace y by y/x^2 and then use formula (2) to get

$$\sum_{k=0}^{n} \binom{n+k}{k} \left(-\frac{y}{x^2}\right)^k x^{-(n+1-k)} L_{n+1-k}(x,\, y) = 1,$$

which reduces easily to the desired identity.

[1] M. Catalani. "Generalized Bivariate Fibonacci Polynomials," http://front.math.ucdavis.edu/math.CO/0211366.

[2] H. W. Gould. "Combinatorial Identities," Morgantown, W. Va., 1972.

Also solved by Paul S. Bruckman and G. C. Greubel.

Lucas Sequences and Sophie Germain Primes

H-622 Proposed by Lawrence Somer, The Catholic University of America, Washington, DC

(Vol. 43, no. 2, May 2005)

Let $u(a,b)$ be the Lucas sequence defined by $u_0 = 0,\ u_1 = 1$ and $u_{n+2} = au_{n+1} - bu_n$ and having discriminant $D(a,b) = a^2 - 4b$, where a and b are integers. Let $\omega(n)$ denote the number of distinct prime divisors of n. Let $c > 1$ be a fixed positive integer. Show that there exist $2^{\omega(c)}$ distinct Lucas sequences $u(a,c^2)$ such that $2p+1 \mid u_p(a,c^2)$ for every Sophie Germain prime p such that $2p+1 \nmid D(a,c^2)$, where $(a,c) = 1$ and $aD(a,c^2) \neq 0$. Recall that p is a Sophie Germain prime if both p and $2p+1$ are primes.

Combined solution by the proposer and the editor

It follows, by the Binet formula, that $u_n(-a,c^2) = (-1)^{n+1}u_n(a,c^2)$. Thus, we can assume without loss of generality that $a > 0$. It suffices to find $2^{\omega(c)-1}$ positive integers a coprime to c such that $2p+1 \mid u_p(a,c^2)$ for all Sophie Germain primes p such that $2p+1 \nmid cD(a,c^2)$.

By a theorem of D. H. Lehmer (see [1, pg. 441]), if $u(a,b)$ is a Lucas sequence and q is an odd prime such that $q \nmid D$ and $(b/q) = 1$, then $q \mid u_{(q-(D/q))/2}$, where (b/q) and (D/q) are Legendre symbols. Thus, if $D(a,c^2) = a^2 - 4c^2$ is equal to a nonzero square m^2, then

$$2p+1 \mid u_p(a,c^2)$$

for all Sophie Germain primes p such that $2p+1 \nmid cD(a,c^2)$. Consequently, it suffices to show that there exist $2^{\omega(c)-1}$ positive integers a coprime to c such that

$$a^2 - 4c^2 = a^2 - (2c)^2 = m^2 \tag{1}$$

for some positive integer m.

Rewriting (1) as $(a-m)(a+m)=(2c)^2=4c^2$, and recalling that m and c are coprime, we see that if we write α for the exact order at which 2 divides $4c^2$, then there exist two coprime odd divisors d_1 and d_2 of c^2 whose product is $c^2/2^{\alpha-2}$ such that $a-m=2^{\alpha-1}d_1$ and $a+m=2d_2$. Note that α is even and that $c=2^{\alpha/2-1}c_1$, where c_1 is odd. It is clear that there are precisely $2^{\omega(c_1)}$ possibilities for an odd divisor d_1 of c_1^2 such that $c_1^2/d_1=d_2$ is coprime to d_1. Further, for each such divisor d_1, we can put

$$a-m=\min\{2^{\alpha-1}d_1,2d_2\}\ \text{ and }\ a+m=\max\{2^{\alpha-1}d_1,d_2\},$$

whose solution is $a=2^{\alpha-2}d_1+d_2$ and $m=|2^{\alpha-2}d_1-d_2|$. Note that we do have $m>0$ since $m=0$ implies $\alpha=2,\ d_1=d_2=1$, therefore $c=1$, which is not allowed. We now note that half of the values of a obtained in this way are distinct for distinct divisors d_1 of c if c is odd, and all of them are distinct if c is even. Indeed, if two such a's are equal, then by (1) and the fact that $m>0$, we get that the corresponding m's are also equal. Thus, the pair of numbers $a-m$ and $a+m$ are the same. When c is odd then d_1 is one of $(a-m)/2$ and $(a+m)/2$. When c is even, then exactly one of $a-m$ and $a+m$ is a multiple of $2^{\alpha-1}$ and d_1 is the corresponding cofactor. Since $\omega(c_1)=\omega(c)-1$ if c is even and $\omega(c_1)=\omega(c)\geq 1$ if $c>1$ is odd, the required conclusion now follows.

[1] D. H. Lehmer. "An Extended Theory of Lucas' Functions," *Ann. of Math.* **31** (1930) : 419–448.

Also solved by Paul S. Bruckman.

A Large Product

H-623 Proposed by José Luis Díaz-Barrero, Barcelona, Spain (Vol. 43, no. 2, May 2005)

Let n be a positive integer. Prove that

$$\prod_{k=1}^{2n-1}(2n-k)^{F_k^2}\leq\left(\frac{F_{2n}}{F_{2n-1}}\right)^{F_{2n-1}F_{2n}}.$$

Solution by H.-J. Seiffert, Berlin, Germany

It is known (see equation (3.32) in [1]), that $F_kF_{k+1}-F_{k-1}F_{k+2}=(-1)^{k+1}$. Also, we have that $F_{k-1}F_{k+2}=(F_{k+1}-F_k)(F_{k+1}+F_k)=F_{k+1}^2-F_k^2$. Now, it is easily verified that

$$kF_k^2=((k+1)F_k-F_{k+1})F_{k+1}-(kF_{k-1}-F_k)F_k+(-1)^k.$$

Summing over $k=1,2,\ldots,2n-1$, and noting the telescoping on the right hand side gives

$$\sum_{k=1}^{2n-1}kF_k^2=(2nF_{2n-1}-F_{2n})F_{2n}. \qquad (1)$$

¿From equation (I_3) in Hoggatt's list, we know that

$$S_n := \sum_{k=1}^{2n-1} F_k^2 = F_{2n-1}F_{2n}. \tag{2}$$

Subtracting (1) from the $2n$-times (2) yields

$$T_n := \sum_{k=1}^{2n-1} (2n-k)F_k^2 = F_{2n}^2. \tag{3}$$

The natural logarithm ln is concave on the interval $(0, +\infty)$. Hence, by Jensen's inequality,

$$\sum_{k=1}^{2n-1} F_k^2 \ln(2n-k) \leq S_n \ln(T_n/S_n).$$

Using (2) and (3), after simplifying and upon exponentiation, one obtains the desired inequality.

[1] A. F. Horadam & Bro. J. M. Mahon. "Pell and Pell-Lucas Polynomials," The Fibonacci Quarterly **23.1** (1985) : 7–20.

Also solved by Said Amghibech, Paul S. Bruckman, Ovidiu Furdui and the proposer.

Limits of Integrals

H-624 Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, MI (Vol. 43, no. 2, May 2005)

Prove that

$$\lim_{n\to\infty} \int_0^1 \frac{1+t^{F_n}}{1+t^{L_n}} dt = 1 \quad \text{and} \quad \lim_{n\to\infty} \int_1^n \frac{1+t^{F_n}}{1+t^{L_n}} dt = 0.$$

Solution by Said Amghibech, St. Foy, Canada

For $t \in [0, 1]$, we have

$$1 \leq \frac{1+t^{F_n}}{1+t^{L_n}} \leq 1 + t^{F_n};$$

thus,

$$1 \leq \int_0^1 \frac{1+t^{F_n}}{1+t^{L_n}} dt \leq \int_0^1 (1+t^{F_n}) dt = 1 + \frac{1}{1+F_n},$$

so

$$\lim_{n\to\infty} \int_0^1 \frac{1+t^{F_n}}{1+t^{L_n}} dt = 1.$$

For $t \in [1, n]$, we have

$$0 \leq \frac{1+t^{F_n}}{1+t^{L_n}} \leq \frac{1}{t^{L_n}} + \frac{1}{t^{L_n - F_n}}.$$

Thus,

$$0 \leq \int_1^n \frac{1+t^{F_n}}{1+t^{L_n}} dt \leq \int_1^\infty \Big(\frac{1}{t^{L_n}} + \frac{1}{t^{L_n-F_n}}\Big) dt = \frac{1}{L_n - 1} + \frac{1}{L_n - F_n - 1},$$

which gives

$$\lim_{n\to\infty} \int_1^n \frac{1+t^{F_n}}{1+t^{L_n}} dt = 0$$

Also solved by Paul S. Bruckman, H.-J. Seiffert and the proposer.

Correction: In H-663 (volume 43.4) the "$A + B = C$" should have been "$A = B + C$".