ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
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PROBLEMS PROPOSED IN THIS ISSUE

H-697 Proposed by N. Gauthier, Kingston, ON

Define $K_0 = 1$ and, for a positive integer n, let K_n represent the sum of the cubes of the first n positive integers. Then define

$$\begin{bmatrix} n \\ k \end{bmatrix}_K = \frac{K_n K_{n-1} \cdots K_{n-k+1}}{K_k K_{k-1} \cdots K_1 K_0}, \qquad \text{for} \quad 0 \le k \le n.$$

a) Show that $\begin{bmatrix} n \\ n-k \end{bmatrix}_K = \begin{bmatrix} n \\ k \end{bmatrix}_K$.

b) Show that $\begin{bmatrix} n \\ k \end{bmatrix}_K = m^2$, where $m = m(n,k)$ is a positive integer.

c) Find a closed form expression for $S_n = \sum_{k \ge 0} m(n,k)$.

H-698 Proposed by Hideyuki Ohtsuka, Saitama, Japan

i) Prove that

$$\left(\sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} = F_{n-1}F_n - \frac{(-1)^n}{3} + O\left(\frac{1}{F_n^2} \right).$$

ii) Is it true that for all nonnegative integers m we have the estimate

$$\left(\sum_{k=n}^{\infty} \frac{1}{F_k F_{k+m}} \right)^{-1} = \sum_{k=1}^{n-1} F_k F_{k+m} + \frac{1}{3} F_{m-2(-1)^n} + O\left(\frac{1}{F_n^2} \right),$$

where the constant implied by the above O might depend on m?

H-699 Proposed by Ovidiu Furdui, Cluj, Romania and Huizeng Qin, Shandong, China

Let $k \geq 0$ be a natural number and let $(x_n)_{n\in\mathbb{N}}$ be the sequence defined by

$$x_n = \sqrt[n]{\Gamma\left(-2k+\frac{1}{2}\right)\Gamma\left(-2k+\frac{1}{3}\right)\cdots\Gamma\left(-2k+\frac{1}{n}\right)}$$
$$-\sqrt[n]{(-1)^{n-1}\Gamma\left(-(2k+1)+\frac{1}{2}\right)\Gamma\left(-(2k+1)+\frac{1}{3}\right)\cdots\Gamma\left(-(2k+1)+\frac{1}{n}\right)},$$

where Γ denotes the classical Gamma function. Find $\lim_{n\to\infty} x_n/n$.

SOLUTIONS

Some Telescoping Series

H-680 Proposed by N. Gauthier, Kingston, ON (Vol. 46, No. 4, November 2008)

For $x \neq 0$ an indeterminate and for an integer $n \geq 0$, consider the generalized Fibonacci and Lucas polynomials $\{f_n\}_n$ and $\{l_n\}_n$, respectively, given by the following recurrences

$$\begin{aligned} f_{n+2} &= xf_{n+1}+f_n \qquad n\geq 0, \quad \text{where} \quad f_0=0,\ f_1=1;\\ l_{n+2} &= xl_{n+1}+l_n \qquad n\geq 0, \quad \text{where} \quad l_0=2,\ l_1=x. \end{aligned}$$

Find closed-form expressions for the following sums:

(a) $\displaystyle\sum_{k=1}^{m}(-1)^{kn}\frac{1}{f_{(k+1)n}f_{kn}}, \quad m,\ n\geq 1;$

(b) $\displaystyle\sum_{k=0}^{m}(-1)^{kn}\frac{1}{l_{(k+1)n}l_{kn}}, \quad m,\ n\geq 0;$

(c) $\displaystyle\sum_{k=1}^{m}(-1)^{kn}\frac{f_{(2k+1)n}}{f^2_{(k+1)n}f^2_{kn}}, \quad m,\ n\geq 1;$

(d) $\displaystyle\sum_{k=0}^{m}(-1)^{kn}\frac{f_{(2k+1)n}}{l^2_{(k+1)n}l^2_{kn}}, \quad m,\ n\geq 0;$

(e) $\displaystyle\sum_{k=0}^{m}(-1)^{kn}\frac{f_{(2k+1)n}[f^2_{(2k+1)n}+f^2_n]}{l^4_{(k+1)n}l^4_{kn}}, \quad m,\ n\geq 0.$

Solution by the proposer

The characteristic equations for the given recurrences are identical and have roots $\alpha = \frac{1}{2}(x+\sqrt{x^2+4})$, $\beta = \frac{1}{2}(x-\sqrt{x^2+4})$, with $\alpha\beta=-1$ and $\alpha+\beta=x$. The Binet form for the terms of the generalized Fibonacci sequence is $f_n=(\alpha^n-\beta^n)/(\alpha-\beta)$ and for the Lucas sequence is $l_n=\alpha^n+\beta^n$. We first prove two results that will simplify the proofs.

1. For integers r and s, we have $f_{r+s}=\frac{1}{2}(f_rl_s+f_sl_r)$.

For the proof, note that

$$\begin{aligned}
f_{r+s} &= \frac{1}{\alpha-\beta}(\alpha^{r+s}-\beta^{r+s})\\
&= \frac{1}{2(\alpha-\beta)}\left((\alpha^{r+s}-\beta^{r+s}+\alpha^r\beta^s-\alpha^s\beta^r)+(\alpha^{r+s}-\beta^{r+s}-\alpha^r\beta^s+\alpha^s\beta^r)\right)\\
&= \frac{1}{2(\alpha-\beta)}\left((\alpha^r-\beta^r)(\alpha^s+\beta^s)+(\alpha^s-\beta^s)(\alpha^r+\beta^r)\right)\\
&= \frac{1}{2}(f_rl_s+f_sl_r),
\end{aligned}$$

which is what we wanted to prove.

2. For integers r and s, we have that $f_{r-s}=\frac{(-1)^s}{2}(f_rl_s-f_sl_r)$.

For the proof,

$$\begin{aligned}
f_{r-s} &= \frac{1}{\alpha-\beta}(\alpha^{r-s}-\beta^{r-s})\\
&= \frac{1}{(\alpha-\beta)}\left(\alpha^r\beta^s(\alpha\beta)^{-s}-\alpha^s\beta^r(\alpha\beta)^{-s}\right)\\
&= \frac{(-1)^s}{2(\alpha-\beta)}\left((\alpha^{r+s}-\beta^{r+s}+\alpha^r\beta^s-\alpha^s\beta^r)-(\alpha^{r+s}-\beta^{r+s}-\alpha^r\beta^s+\alpha^s\beta^r)\right)\\
&= \frac{(-1)^s}{2(\alpha-\beta)}\left((\alpha^r-\beta^r)(\alpha^s+\beta^s)-(\alpha^s-\beta^s)(\alpha^r+\beta^r)\right)\\
&= \frac{(-1)^s}{2}(f_rl_s-f_sl_r),
\end{aligned}$$

which is what we wanted to prove.

Now, with (n,k) integers, put $r:=n(k+1)$ and $s:=nk$ in the above formulas and rearrange the results in either one of the following forms, by dividing by $f_{n(k+1)}f_{nk}$ or by $l_{n(k+1)}l_{nk}$, as the case may be, to get that:

(1a) $\dfrac{f_{n(2k+1)}}{f_{n(k+1)}f_{nk}}=\dfrac{1}{2}\left(\dfrac{l_{nk}}{f_{nk}}+\dfrac{l_{n(k+1)}}{f_{n(k+1)}}\right),\ n\ge 1,\ k\ge 1;$

(1b) $\dfrac{f_{n(2k+1)}}{l_{n(k+1)}l_{nk}}=\dfrac{1}{2}\left(\dfrac{f_{n(k+1)}}{l_{n(k+1)}}+\dfrac{f_{nk}}{l_{nk}}\right),\ n\ge 0,\ k\ge 0;$

(2a) $\dfrac{(-1)^{nk}f_n}{f_{n(k+1)}f_{nk}}=\dfrac{1}{2}\left(\dfrac{l_{nk}}{f_{nk}}-\dfrac{l_{n(k+1)}}{f_{n(k+1)}}\right),\ n\ge 1,\ k\ge 1;$

(2b) $\dfrac{(-1)^{nk}f_n}{l_{n(k+1)}l_{nk}}=\dfrac{1}{2}\left(\dfrac{f_{n(k+1)}}{l_{n(k+1)}}-\dfrac{f_{nk}}{l_{nk}}\right),\ n\ge 0,\ k\ge 0.$

To find the sought closed forms, we first invoke (2a) and sum the resulting telescoping series. This gives the desired closed form for sum (a) upon division of (2a) by f_n:

Closed form for (a):

$$\sum_{k=1}^{m}(-1)^{nk}\frac{1}{f_{n(k+1)}f_{nk}}=\frac{1}{2f_n}\sum_{k=1}^{m}\left(\frac{l_{nk}}{f_{nk}}-\frac{l_{n(k+1)}}{f_{n(k+1)}}\right)=\frac{1}{2f_n}\left(\frac{l_n}{f_n}-\frac{l_{n(m+1)}}{f_{n(m+1)}}\right),\quad m\ge 1,\ n\ge 1.$$

We proceed similarly for sum (b) and get, upon division of (2b) by f_n, that:

Closed form for (b):

$$\sum_{k=0}^{m}(-1)^{nk}\frac{1}{l_{n(k+1)}l_{nk}}=\frac{1}{2f_n}\sum_{k=0}^{m}\left(\frac{f_{n(k+1)}}{l_{n(k+1)}}-\frac{f_{nk}}{l_{nk}}\right)=\frac{f_{n(m+1)}}{2f_nl_{n(m+1)}},\quad m\geq 0,\ n\geq 1.$$

To proceed further, form the product of equation (1a) by equation (2a) and get that:

$$(3)\quad (-1)^{nk}\frac{f_nf_{n(2k+1)}}{f^2_{n(k+1)}f^2_{nk}}=\frac{1}{4}\left(\frac{l^2_{n(k+1)}}{f^2_{n(k+1)}}-\frac{l^2_{nk}}{f^2_{nk}}\right).$$

Dividing this result by f_n then gives the summand in (c) and the resulting sum telescopes:

Closed form for (c):

$$\sum_{k=1}^{m}(-1)^{nk}\frac{f_{n(2k+1)}}{f^2_{n(k+1)}f^2_{nk}}=\frac{1}{4f_n}\left(\frac{l^2_{n(m+1)}}{f^2_{n(m+1)}}-\frac{l^2_n}{f^2_n}\right),\quad m\geq 1,\ n\geq 1.$$

Similarly, form the product of (1b) by (2b) and get:

$$(4)\quad (-1)^{nk}\frac{f_nf_{n(2k+1)}}{l^2_{n(k+1)}l^2_{nk}}=\frac{1}{4}\left(\frac{f^2_{n(k+1)}}{l^2_{n(k+1)}}-\frac{f^2_{nk}}{l^2_{nk}}\right).$$

This gives the summand of sum (d) upon division by f_n and the sum collapses to give:

Closed form for (d):

$$\sum_{k=0}^{m}(-1)^{nk}\frac{f_{n(2k+1)}}{l^2_{n(k+1)}l^2_{nk}}=\frac{f^2_{n(m+1)}}{4f_nl^2_{n(m+1)}},\quad m\geq 0,\ n\geq 1.$$

Next, take the square of equation (1b) and add the result to the square of equation (2b). This gives:

$$(5)\quad \frac{f^2_{n(2k+1)}+f^2_n}{l^2_{n(k+1)}l^2_{nk}}=\frac{1}{4}\left(\left(\frac{f_{n(k+1)}}{l_{n(k+1)}}+\frac{f_{nk}}{l_{nk}}\right)^2+\left(\frac{f_{n(k+1)}}{l_{n(k+1)}}-\frac{f_{nk}}{l_{nk}}\right)^2\right)=\frac{1}{2}\left(\frac{f^2_{n(k+1)}}{l^2_{n(k+1)}}+\frac{f^2_{nk}}{l^2_{nk}}\right).$$

Multiplication of this result by (4) then gives:

$$(6)\quad (-1)^{nk}\frac{f_nf_{n(2k+1)}(f^2_{n(k+1)}+f^2_n)}{l^4_{n(k+1)}l^4_{nk}}=\frac{1}{8}\left(\frac{f^4_{n(k+1)}}{l^4_{n(k+1)}}-\frac{f^4_{nk}}{l^4_{nk}}\right).$$

This gives the summand in sum (e) upon division by f_n and we get the desired result due to the collapsing of the series:

Closed form for (e):

$$\sum_{k=0}^{m}(-1)^{nk}\frac{f_{n(2k+1)}(f^2_{n(k+1)}+f^2_n)}{l^4_{n(k+1)}l^4_{nk}}=\frac{f^4_{n(m+1)}}{8f_nl^4_{n(m+1)}},\quad m\geq 0,\ n\geq 1.$$

Also solved by Paul S. Bruckman.

Integral Power Binomial Weighted Sums of Generalized Fibonacci Polynomials

H-681 Proposed by N. Gauthier, Kingston, ON
(Vol. 47, No. 1, February 2009/2010)

For a real variable $z \neq 0$ consider the sets of generalized Fibonacci and Lucas polynomials, $\{f_n = f_n(z) : n \in \mathbb{Z}\}$ and $\{l_n = l_n(z) : n \in \mathbb{Z}\}$, given by the recurrences

$$f_{n+2} = zf_{n+1} + f_n, \quad \text{and} \quad l_{n+2} = zl_{n+1} + l_n, \quad \text{for all} \quad n \in \mathbb{Z},$$

with $f_0 = 0,\ f_1 = 1,\ l_0 = 2,\ l_1 = z$. Note that $f_{-n} = (-1)^{n+1} f_n$ and $l_{-n} = (-1)^n l_n$. Let r be a nonnegative integer and $p,\ q$ be positive integers.

(a) Prove that

$$\sum_{k\geq 0} (-1)^k k \binom{r}{k} f_p^k f_{p+q}^{r-k} l_{qk} = (-1)^{q+1} r f_p f_q^{r-1} l_{pr-(p+q)}.$$

(b) Find a general formula for $\sum_{k\geq 0} (-1)^k k^m \binom{r}{k} f_p^k f_{p+q}^{r-k} l_{qk}$ for any nonnegative integer m.

Solution by the proposer

The characteristic equations for the given recurrences are identical and have roots $\alpha = \frac{1}{2}(z + \sqrt{z^2+4})$, $\beta = \frac{1}{2}(z - \sqrt{z^2+4})$, with $\alpha\beta = -1$ and $\alpha + \beta = z$. The Binet form for the terms of the generalized Fibonacci sequence is $f_n = (\alpha^n - \beta^n)/(\alpha - \beta)$ and for the Lucas sequence is $l_n = \alpha^n + \beta^n$.

To prove identity (a), we use the following lemmas.

Lemma 1. *For x a variable and r a nonnegative integer, we have*

$$\sum_{k\geq 0} (-1)^{r-k} k \binom{r}{k} (1+x)^k = r x^{r-1}(1+x).$$

Proof. First note that

$$\sum_{k\geq 0} (-1)^{r-k} \binom{r}{k} (1+x)^k = x^r,$$

which follows from the binomial expansion of $x^r = (-1 + (1+x))^r$ in powers of $(1+x)$. Then apply the differential operator $(1+x)\dfrac{d}{dx}$ to this result and get that

$$\sum_{k\geq 0} (-1)^{r-k} k \binom{r}{k} (1+x)^k = r x^{r-1}(1+x); \quad r \geq 0,$$

which proves Lemma 1. □

Lemma 2. *For positive integers (p,q), the solution of the following simultaneous equations*

$$1 + u\alpha^p = w\alpha^{-q}, \qquad 1 + u\beta^p = w\beta^{-q},$$

for the unknowns u and w is:

$$u = -\frac{f_q}{f_{p+q}}, \qquad w = (-1)^q \frac{f_p}{f_{p+q}}.$$

Proof. One can get at once that $\alpha^q + u\alpha^{p+q} = w = \beta^q + u\beta^{p+q}$. Hence, since $p+q \neq 0$, we get that

$$u = \frac{\beta^q - \alpha^q}{\alpha^{p+q} - \beta^{p+q}} = -\frac{f_q}{f_{p+q}}.$$

Similarly, one can see that $-\alpha^{-p} + w\alpha^{-(p+q)} = u = -\beta^{-p} + w\beta^{-(p+q)}$. Hence, we get using the Binet formula for the Fibonacci polynomials that

$$w = \frac{\alpha^{-p} - \beta^{-p}}{\alpha^{-(p+q)} - \beta^{-(p+q)}} = \frac{f_{-p}}{f_{-(p+q)}} = \frac{(-1)^{p+1} f_p}{(-1)^{p+q+1} f_{p+q}} = (-1)^q \frac{f_p}{f_{p+q}},$$

which proves Lemma 2. □

We now prove summation formula (a). To do so, first note that $\alpha\beta = -1$ implies that $\alpha^{-q} = (-1)^q \beta^q$. We use Lemma 1 and 2 with

$$x = u\alpha^p = -\frac{f_q}{f_{p+q}}\alpha^p, \qquad (1+x) = w\alpha^{-q} = (-1)^q w \beta^q = \frac{f_p}{f_{p+q}}\beta^q,$$

to get that

$$\sum_{k\geq 0} (-1)^{r-k} k \binom{r}{k} \left(\frac{f_p}{f_{p+q}}\beta^q\right)^k = r\left(-\frac{f_q}{f_{p+q}}\alpha^p\right)^{r-1}\left(\frac{f_p}{f_{p+q}}\beta^q\right) = (-1)^{q+r-1} r \frac{f_p f_q^{r-1}}{f_{p+q}^r}\alpha^{pr-(p+q)}.$$

Repeating the exercise with

$$x = -\frac{f_q}{f_{p+q}}\beta^p \qquad \text{and} \qquad (1+x) = \frac{f_p}{f_{p+q}}\alpha^q,$$

gives that

$$\sum_{k\geq 0} (-1)^{r-k} k \binom{r}{k} \frac{f_p^k}{f_{p+q}^k}\alpha^{qk} = (-1)^{q+r-1} r \frac{f_p f_q^{r-1}}{f_{p+q}^r}\beta^{pr-(p+q)}.$$

Finally, add these last two results together and multiply the resulting equation by $(-1)^r f_{p+q}^r$ to get identity (a):

$$\sum_{k\geq 0} (-1)^k k \binom{r}{k} f_p^k f_{p+q}^{r-k} l_{qk} = (-1)^{q+1} r f_p f_q^{r-1} l_{pr-(q+1)}, \quad r \geq 0.$$

To generalize the problem as requested in part (b), we will use the following lemma.

Lemma 3. *For x an arbitrary variable and for an integer $r \neq 0$, we have*

$$\sum_{k\geq 0} (-1)^{r-k} k^m \binom{r}{k} (1+x)^k = \sum_{n=0}^{m} (r)_n S_n^{(m)} x^{r-n} (1+x)^n,$$

where $\{S_n^{(m)} : 0 \leq m, 0 \leq n \leq m\}$ is the augmented set of Stirling numbers of the second kind, including the $n = 0$ elements, $S_0^{(m)} = \delta_{m,0}$. Also, by definition, for $n \geq 1$, $(r)_n = r(r-1)\cdots(r-n+1)$ and for $n = 0$, $(r)_0 = 1$.

Proof. For $m \geq 0$, consider the differential operator $\left((1+x)\dfrac{d}{dx}\right)^m$ and apply it to the formula

$$\sum_{k\geq 0} (-1)^{r-k} \binom{r}{k} (1+x)^k = x^r$$

(see Lemma 1). After noting that $\left((1+x)\frac{d}{dx}\right)^m (1+x)^k = k^m(1+x)^k$ as well as the fact that $\left((1+x)\frac{d}{dx}\right)^m x^r$ generates an $m+1$-term expansion in $\{x^{r-n}(1+x)^n : 0 \le n \le m\}$, we claim that the following holds for nonnegative r, m:

$$\sum_{k\ge 0}(-1)^{r-k}k^m\binom{r}{k}(1+x)^k = \sum_{n=0}^{m}(r)_n a_n^{(m)} x^{r-n}(1+x)^n.$$

The unknown coefficients, $\{a_n^{(m)} : 0 \le m, 0 \le n \le m\}$, are to be determined by solving the following linear recurrence:

$$a_n^{(m+1)} = na_n^{(m)} + a_{n-1}^{(m)};\quad a_0^{(0)} = 1,\quad a_{-1}^{(m)} = a_{m+1}^{(m)} = 0.$$

To prove the above claim, note that it is true for $m = 0$ if we convene that $k^0 = 1$ for all $k \ge 0$. So, assuming that the above formula is true for m, consider

$$\left((1+x)\frac{d}{dx}\right)^{m+1} x^r = (1+x)\frac{d}{dx}\left(\left((1+x)\frac{d}{dx}\right)^m x^r\right).$$

Upon invoking the above expressions for $\left((1+x)\frac{d}{dx}\right)^{m+1} x^r$ and of $\left((1+x)\frac{d}{dx}\right)^m x^r$ in powers of $(1+x)/x$, we get that

$$\begin{aligned}
\sum_{n=0}^{m+1}(r)_n a_n^{(m+1)} x^{r-n}(1+x)^n &= (1+x)\frac{d}{dx}\sum_{n=0}^{m}(r)_n a_n^{(m)} x^{r-n}(1+x)^n \\
&= \sum_{n=0}^{m}(r)_n a_n^{(m)}\left((r-n)x^{r-n-1}(1+x)^{n+1} + nx^{r-n}(1+x)^n\right) \\
&= \sum_{n=0}^{m}(r)_{n+1} a_n^{(m)} x^{r-(n+1)}(1+x)^{n+1} + \sum_{n=0}^{m} n(r)_n a_n^{(m)} x^{r-n}(1+x)^n \\
&= \sum_{n=0}^{m+1}(r)_n\left(a_{n-1}^{(m)} + na_n^{(m)}\right) x^{r-n}(1+x)^n.
\end{aligned}$$

To go from the penultimate line to the last one above, we shifted the summation index in the first sum by one unit. Then we defined $a_{-1}^{(m)} = 0$, $a_{m+1}^{(m)} = 0$ and extended the limits of both sums from 0 to $m+1$. This result then gives the recurrence for the unknown coefficients, which is the recurrence for the augmented Stirling numbers of the second kind, $S_n^{(m)}$. We therefore conclude that $\{a_n^{(m)} = S_n^{(m)} : 0 \le m,\ 0 \le n \le m\}$ and Lemma 3 is proved. □

Now, to obtain the generalization requested in (b) of the problem statement, we invoke Lemmas 2 and 3 and proceed as we did to prove identity (a). We then get the following two equations:

$$\sum_{k\geq 0}(-1)^{r-k}k^m\binom{r}{k}\frac{f_p^k}{f_{p+q}^k}\beta^{qk} = \sum_{n=0}^{m}(r)_n S_n^{(m)}\left(-\frac{f_q}{f_{p+q}}\alpha^p\right)^{r-n}\left(\frac{f_p}{f_{p+q}}\beta^q\right)^n$$

$$= \sum_{n=0}^{m}(-1)^{(q+1)n+r}(r)_n S_n^{(m)}\frac{f_p^n f_q^{r-n}}{f_{p+q}^r}\alpha^{pr-(p+q)n};$$

$$\sum_{k\geq 0}(-1)^{r-k}k^m\binom{r}{k}\frac{f_p^k}{f_{p+q}^k}\alpha^{qk} = \sum_{n=0}^{m}(-1)^{(q+1)n+r}(r)_n S_n^{(m)}\frac{f_p^n f_q^{r-n}}{f_{p+q}^r}\beta^{pr-(p+q)n}.$$

Adding together these two equations and multiplying the result by $(-1)^r f_{p+q}^r$ then gives the sought generalization

$$\sum_{k\geq 0}(-1)^k k^m\binom{r}{k}f_p^k f_{p+q}^{r-k}l_{qk} = \sum_{n=0}^{m}(-1)^{(q+1)n}(r)_n S_n^{(m)} f_p^n f_q^{r-n} l_{pr-(p+q)n}.$$

This result agrees with the identity in (a) when $m = 1$ since $S_0^{(1)} = 0$, $S_1^{(1)} = 1$ and $(r)_1 = r$.

Also solved by Paul S. Bruckman and Kenneth Davenport.