ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
FLORIAN LUCA

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, CCM, UNAM, AP. POSTAL 61-3 (XANGARI), CP 58 089, MORELIA, MICHOACAN, MEXICO, or by e-mail at fluca@matmor.unam.mx as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-730 Proposed by N. Gauthier, Kingston, ON

Let $\lfloor x \rfloor$ be the largest integer less than or equal to x and let $\varepsilon_n = (1+(-1)^n)/2$. Then, with P_n the nth Pell number prove the following identities:

(a) $\displaystyle\sum_{k\ge 0} \frac{1}{25^k}\binom{n-2k}{2k} = \frac{1}{5^{n/2}6}\left[\varepsilon_n(L_{2n+2}+3L_{n+1}) + (1-\varepsilon_n)\sqrt{5}(F_{2n+2}+3F_{n+1})\right];$

(b) $\displaystyle\sum_{k\ge 0} \frac{1}{16^k}\binom{n-1-2k}{2k} = \frac{1}{2^n}[P_n+n];$

(c)

$$\sum_{k=0}^{\lfloor (n-1)/4 \rfloor} \frac{1}{25^k(n-4k)}\binom{n-1-2k}{2k} = \frac{1}{5^{n/2}n}\Big[\varepsilon_n(L_{2n}+L_n-2(1+(-1)^{n/2})) + (1-\varepsilon_n)\sqrt{5}(F_{2n}+F_n)\Big];$$

(d)

$$\sum_{k\ge 1} \frac{k}{5^k}\binom{n-1-k}{k} = \frac{1}{5^{n/2}54}\Big[\varepsilon_n((45n-20)F_{2n}-15nL_{2n}) + (1-\varepsilon_n)\sqrt{5}((9n-4)L_{2n}-15nF_{2n})\Big].$$

H-731 Proposed by Anastasios Kotronis, Athens, Greece

Show that

$$f(x) := \sum_{n=1}^{\infty} \frac{n\cosh(nx)}{\sinh(n\pi)} = \frac{1}{(\pi-x)^2} + \frac{3\pi-12}{12\pi} + O\left((\pi-x)\right) \qquad \text{as} \quad x\to\pi^-.$$

H-732 Proposed by N. Gauthier, Kingston, ON

In the following, C_k is the kth Catalan number with the convention that $C_k = 0$ if $k < 0$.

(1) For nonnegative integers m, n let

$$c_m(n) = \sum_{k \geq 0} (-1)^k \binom{n-k}{k} C_{n-m-k}.$$

Find a closed form for $c_m(n)$.

(2) For nonnegative integers m, n let

$$G_m(n) = \sum_{k \geq 0} (-1)^k \binom{n-k}{k} \binom{2(n-m-k)}{n-m-k}.$$

(a) Show that $G_m(n) = 0$ for $0 \leq n \leq m-1$.

(b) Find a closed form for $G_m(n)$ if $n \geq 2m$.

(c) Show that $G_m(n+m)$ is a polynomial of degree n in m and express the polynomial coefficients as a ratio of two determinants.

H-733 Proposed by H. Ohtsuka, Saitama, Japan

Define the sequence $\{H_n\}_{n \geq -1}$ given by $H_{-1} = \mathbf{i},\ H_0 = 0,\ H_{n+2} = H_{n+1} - \mathbf{i}H_n$ for $n \geq -1$, where $\mathbf{i} = \sqrt{-1}$. Find an explicit formula for $\sum_{k=1}^{n} H_k^4$.

H-734 Proposed by H. Ohtsuka, Saitama, Japan

For $n \geq 3$ find closed form expressions for

$$\left\lfloor \left(1 - \prod_{k=n}^{\infty}\left(1 - \frac{1}{F_k}\right)\right)^{-1} \right\rfloor \quad \text{and} \quad \left\lfloor \left(1 - \prod_{k=n}^{\infty}\left(1 - \frac{1}{F_k^2}\right)\right)^{-1} \right\rfloor.$$

Here, $\lfloor x \rfloor$ be the largest integer less than or equal to x.

SOLUTIONS

A Sum Yielding Pell Numbers

H-704 Proposed by Paul S. Bruckman, Nanaimo, BC (Vol. 48, No. 3, August 2011)

Prove the following identity:

$$\sum_{k=0}^{\lfloor n/4 \rfloor} \binom{n-2k}{2k} 2^{n+1-4k} = P_{n+1} + n + 1,$$

where $\{P_n\}_{n \geq 0}$ is the ordinary Pell sequence.

Solution by Zbigniew Jakubczyk, Warsaw, Poland.

The generating function of this sequence is

$$\begin{aligned}\sum_{n=0}^{\infty} s_n z^n &= \sum_{n=0}^{\infty}\sum_{4k\le n}\binom{n-2k}{2k}2^{n+1-4k}z^n = \sum_{m=0}^{\infty}\sum_{4k\le m+2k}\binom{m}{2k}2^{m+1-2k}z^{m+2k}\\ &= \sum_{m=0}^{\infty}\sum_{2k\le m}\binom{m}{2k}2(2z)^{m-2k}z^{4k} = 2\sum_{m=0}^{\infty}\sum_{2k\le m}\binom{m}{2k}(2z)^{m-2k}(z^2)^{2k}\\ &= \sum_{m=0}^{\infty}\left((2z+z^2)^m+(2z-z^2)^m\right) = \frac{1}{1-2z-z^2}+\frac{1}{1-2z+z^2}\\ &= \frac{1}{z}\left(\frac{z}{1-2z-z^2}\right)+\frac{1}{(1-z)^2} = \frac{1}{z}\left(\frac{z}{1-2z-z^2}\right)+\left(\frac{1}{1-z}\right)'.\end{aligned}$$

Since the generating function of the Pell sequence is $z/(1-2z-z^2)$, we find

$$\begin{aligned}\sum_{n=0}^{\infty} s_n z^n &= \frac{1}{z}\sum_{n=0}^{\infty}P_n z^n + \left(\sum_{n\ge 0} z^n\right)'\\ &= \sum_{n=0}^{\infty}P_{n+1}z^n + \sum_{n=0}^{\infty}(n+1)z^n\\ &= \sum_{n\ge 0}(P_{n+1}+n+1)z^n.\end{aligned}$$

So, $s_n = P_{n+1}+n+1$.

Also solved by Annita Davis & Cecil Rousseau, Kenneth Davenport, Ángel Plaza & Sergio Falcón, and the proposer.

A Recurrence for Sums of Binomial Coefficients

H-705 Proposed by Paul S. Bruckman, Nanaimo, Canada (Vol. 48, No. 3, August 2011)

Define the following sum

$$S_n(a,b) = \sum_{k=0}^{\lfloor (n-b)/a\rfloor}\binom{n}{ak+b},$$

where $n,\ a$ and b are integers with $0\le b<a\le n.$ Prove the following relation: $S_{am+2b}(a,b) = 2S_{am+2b-1}(a,b),\ m=1,2,\ldots.$

Solution by Ángel Plaza, Las Palmas, Spain.

We have to prove the following identity

$$S_{am+2b}(a,b) = 2S_{am+2b-1}(a,b),$$

or, equivalently,

$$\sum_{k=0}^{\lfloor (am+b)/a\rfloor}\binom{am+2b}{ak+b} = 2\sum_{k=0}^{\lfloor (am+b-1)/a\rfloor}\binom{am+2b-1}{ak+b}.$$

Since $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$, it is enough to prove that

$$\sum_{k\geq 0} \binom{am+2b-1}{ak+b} = \sum_{k\geq 0} \binom{am+2b-1}{ak+b-1},$$

where it is supposed that the binomial terms are zero if the lower index is negative or greater than the upper index. On the other hand, since $\binom{am+2b-1}{ak+b} = \binom{am+2b-1}{a(m-k)+b-1}$, the conclusion follows.

Also solved by the proposer.

<u>**Harmonic Sums and the Prime Counting Function**</u>

<u>**H-706**</u> **Proposed by Paul S. Bruckman, Nanaimo, Canada (Vol. 48, No. 3, August 2011)**

Define the following sum:

$$S_n = \frac{1}{2}\left(\sum_{k=n+1}^{3n} \frac{1}{k^2-n^2}\right)^{-1}.$$

Show that $S(n) \sim \pi(n)$ as $n \to \infty$, where $\pi(n)$ is the counting function of the primes $p \leq n$.

Solution by Ángel Plaza, Las Palmas, Spain.

Since $\dfrac{1}{k^2-n^2} = \dfrac{1/(2n)}{k-n} - \dfrac{1/(2n)}{k+n}$, then

$$\begin{aligned}\sum_{k=n+1}^{3n} \frac{1}{k^2-n^2} &= \frac{1}{2n}\sum_{k=n+1}^{3n} \frac{1}{k-n} - \frac{1}{2n}\sum_{k=n+1}^{3n} \frac{1}{k+n} \\ &= \frac{1}{2n}\left(H_{2n} - H_{4n} + H_{2n}\right) \\ &= \frac{1}{2n}\left(2H_{2n} - H_{4n}\right),\end{aligned}$$

where $H_n = \sum_{k=1}^{n} \frac{1}{k}$ is the nth harmonic number. Therefore, $S_n = \dfrac{n}{2H_{2n} - H_{4n}}$.

For n large enough, the following bounds will be of help [1]:

$$\frac{n}{\ln n + 2} < \pi(n) < \frac{n}{\ln n - 4}.$$

Therefore, in order to prove that $S_n \sim \pi(n)$ it is enough to see that

$$\frac{1}{\ln n + 2} < \frac{1}{2H_{2n} - H_{4n}} < \frac{1}{\ln n - 4},$$

or, equivalently,

$$\ln n - 4 < 2H_{2n} - H_{4n} < \ln n + 2.$$

But these last inequalities hold trivially, since $\dfrac{1}{2(n+1)} < H_n - \ln n - \gamma < \dfrac{1}{2n}$.

References

[1] B. Rosser, *Explicit bounds for some functions of prime numbers*, American Journal of Mathematics, **63.1** (1941), 211–232.

Also solved by Anastasios Kotronis and the proposer.

An Identity with Continued Fractions

H-707 Proposed by Paul S. Bruckman, Nanaimo, Canada (Vol. 48, No. 3, August 2011)

Write $[P] = [a_1, a_2, \ldots, a_n]$, where $a_k = a_{n+1-k}$, $k = 1, 2, \ldots, n$; then $[P]$ is a palindromic simple continued fraction (scf); here, the a_k's are positive integers. Also, write $[0, P^*] = [0, a_1, \ldots, a_{n-1}]$. Finally, let $[\overline{P}]$ denote the infinite periodic scf $[P, P, P, \ldots]$. Prove the following: $[\overline{P}] - [0, \overline{P}] = [P] - [0, P^*]$.

Solution by the proposer

Let $x = [\overline{P}] = [P, x]$. Then $x = (Px + 1)/x$, so

$$x^2 - PX - 1 = 0. \tag{1}$$

Note that $P = x - x^{-1} = [\overline{P}] - [0, \overline{P}]$. Consider the last two convergence of $[P]$, namely say r/s and t/u, say. By the so-called Mirror formula, $[P] = t/u$ and $[0, P^*] = s/u$. Thus, $[P] - [0, P^*] = (t - s)/u$. On the other hand, the last three convergents of $x = [\overline{P}] = [P, x]$ are u/s, t/u and $(tx + u)/(ux + s)$. Then $ux^2 + sx = tx + u$, or

$$x^2 - \left(\frac{t-s}{u}\right) x - 1 = 0. \tag{2}$$

Comparing (1) and (2), we get $P = (t - s)/u = [\overline{P}] - [0, \overline{P}] = [P] - [0, P^*]$, which is what we wanted to prove.

Note: If $n = 1$, then $P = a_1$ and $[0, P^*] = 0$. If $n = 2$, we have essentially the same case as when $n = 1$. Hence, we may therefore assume that $n = 1$ or $n \geq 3$.

An Inequality with Fibonacci Numbers

H-708 Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain (Vol. 48, No. 4, November 2011)

Let n be a positive integer. Prove that

$$\left(\frac{n}{F_n^2 + F_{n+1}^2}\right)^2 + \left(\frac{1}{4n^2}\prod_{k=1}^{n}\frac{1}{F_k^4}\right)\left(\sum_{k=1}^{n}(F_k^4 - 1)^{1/2}\right)^2 \leq \frac{1}{4}.$$

Solution by Hideyuki Ohtsuka, Saitama, Japan.

Let

$$An = \left(\frac{n}{F_n^2 + F_{n+1}^2}\right)^2 + \left(\frac{1}{4n^2}\prod_{k=1}^{n}\frac{1}{F_k^4}\right)\left(\sum_{k=1}^{n}(F_k^4 - 1)^{1/2}\right)^2.$$

We have

$$A_1 = \frac{1}{4}, \quad A_2 = \frac{4}{25} < \frac{1}{4}, \quad A_3 = \frac{2573}{32448} < \frac{1}{4},$$

and

$$A_4 = \frac{4}{289} + \frac{(4\sqrt{5}+\sqrt{15})^2}{5184} < \frac{1}{4}.$$

For $n \geq 5$, we have

$$\begin{aligned} A_n &< \left(\frac{n}{2n^2}\right)^2 + \frac{1}{4n^2F_n^4}\left(\sum_{k=1}^{n} F_k^2\right)^2 \qquad (\text{by} \quad F_{n+1} > F_n \geq n \geq 5) \\ &= \frac{1}{4n^2} + \frac{(F_nF_{n+1})^2}{4n^2F_n^4} = \frac{1}{4n^2} + \frac{F_{n+1}^2}{4n^2F_n^2} \\ &< \frac{1}{4n^2} + \frac{2F_n)^2}{4n^2F_n^2} = \frac{5}{4n^2} \leq \frac{1}{20} < \frac{1}{4}. \end{aligned}$$

Also solved by Paul Bruckman and the proposer.

Late Acknowledgement: Kenneth Davenport has solved **H–694** and **H–701**.