ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, IMATE, UNAM, AP. POSTAL 61-3 (XANGARI), CP 58 089, MORELIA, MICHOACAN, MEXICO, or by e-mail at fluca@matmor.unam.mx as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-639 Proposed by H.-J. Seiffert, Berlin, Germany

The sequences of the Fibonacci and Lucas polynomials are defined by

$$F_0(x)=0,\ F_1(x)=1,\ \text{and } F_{n+1}(x)=xF_n(x)+F_{n-1}(x) \text{ for } n\geq 1,$$

$$L_0(x)=2,\ L_1(x)=x,\ \text{and } L_{n+1}(x)=xL_n(x)+L_{n-1}(x) \text{ for } n\geq 1,$$

respectively. Prove that, for all non-zero complex numbers x and all positive integers n,

(a)

$$\sum_{k=0}^{2n-1}\binom{4n-1-k}{k}2^{4n-1-2k}x^kF_k(x)=x^{2n-1}L_{2n-1}(x)F_{2n}(4/x),$$

(b)

$$\sum_{k=0}^{2n-1}\binom{4n-1-k}{k}2^{4n-1-2k}x^kL_k(x)=x^{2n-1}(x^2+4)F_{2n-1}(x)F_{2n}(4/x),$$

(c)

$$\sum_{k=0}^{2n}\binom{4n+1-k}{k}2^{4n+2-2k}x^kF_k(x)=x^{2n+1}F_{2n}(x)L_{2n+1}(4/x),$$

(d)

$$\sum_{k=0}^{2n}\binom{4n+1-k}{k}2^{4n+2-2k}x^kL_k(x)=x^{2n+1}L_{2n}(x)L_{2n+1}(4/x).$$

H-640 **Proposed by Jayantibhai M. Patel, Ahmedabad, India**

For any positive integer $n \geq 2$, prove that the value of the following determinant

$$\begin{vmatrix} -L_{2n} & F_{2n} & L_n^2 & 2F_{2n} & L_n^2 \\ F_{2n} & -3(3F_n^2+2(-1)^n) & F_{2n} & 2F_n^2 & F_{2n} \\ L_n^2 & F_{2n} & -L_{2n} & 2F_{2n} & L_n^2 \\ 2F_{2n} & 2F_n^2 & 2F_{2n} & -6F_{n+1}F_{n-1} & 2F_{2n} \\ L_n^2 & F_{2n} & L_n^2 & 2F_{2n} & -L_{2n} \end{vmatrix}$$

is $(L_n^2 + L_{2n})^5$.

H-641 **Proposed by Emeric Deutsch, Polytechnic University, Brooklyn, NY**

A composition of n is an ordered sequence of positive integers having sum equal to n. The terms of the sequence are called the parts of n. It is known that the number of compositions of n is 2^{n-1} and the number of compositions of n with exactly k parts is equal to $\binom{n-1}{k-1}$. Here, we consider a slightly modified concept assuming that there are two kinds of 1. Find:

(i) The number a_n of compositions of n (for example, $a_2 = 5$ because we have (2), $(1,1)$, $(1,1')$, $(1',1)$, and $(1',1')$);

(ii) the number $c_{n,k}$ of compositions of n with exactly k parts (for example, $c_{4,2} = 5$ because we have $(1,3)$, $(1',3)$, $(3,1)$, $(3,1')$, and $(2,2)$).

H-642 **Walther Janous, Innsbruck, Austria**

Determine the limit

$$\lim_{n\to\infty}\Big(\frac{L_{n+2}^2}{F_{n+2}} - \sum_{k=1}^{n}\frac{L_k^2}{F_k}\Big).$$

SOLUTIONS

Bounding ratios of Fibonacci numbers

H-625 **Proposed by Russel Jay Hendel, Townson University, MD**
(Vol. 3, no. 2, May 2005)

For an integer $m > 0$ let K_m be the smallest positive integer such that $F_{n+m} < K_m F_n$ holds for all large n. For example, $K_1 = 2$ because $F_n < F_{n+1} < 2F_n$ holds for all large n. Provide an explicit formula for K_m.

Solution by H.-J. Seiffert

It is known (see equations (3.22) and (3.24) in [1]) that, for all integers m and n,

$$F_{n+m} + (-1)^m F_{n-m} = L_m F_n. \tag{1}$$

We shall prove that, for $m > 0$,

$$K_m = \begin{cases} L_m + 1, & \text{if } m \text{ is odd,} \\ L_m\ , & \text{if } m \text{ is even.} \end{cases}$$

Suppose that m is odd. If $n > m$, then, by (1),

$$L_m F_n = F_{n+m} - F_{n-m} < F_{n+m} < F_{n+m} + F_n - F_{n-m} = (L_m + 1)F_n.$$

Let m be even. If $n > m$, then, by (1) again,

$$(L_m - 1)F_n = F_{n+m} - (F_n - F_{n-m}) < F_{n+m} < F_{n+m} + F_{n-m} = L_m F_n.$$

This completes the proof of the above statement.

[1] A. F. Horadam & Bro. J. M. Mahon. "Pell and Pell-Lucas Polynomials," The Fibonacci Quarterly **23.1** (1985) : 7–20.

Also solved by Paul S. Bruckman and the proposer.

Pell numbers and Fibonacci polynomials

H-626 Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 43, no. 2, May 2005)

The Fibonacci polynomials are defined by $F_0(x) = 0$, $F_1(x) = 1$, and $F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$ for $n \geq 0$. Let n be a positive integer.

a. Prove that, for all complex numbers x,

$$F_{n+1}(x) + iF_n(x) = 4^{-n} \sum_{k=0}^{n} \binom{2n+1}{2k+1} (x-2i)^k (x+2i)^{n-k}, \quad \text{where } i = \sqrt{-1}.$$

b. Deduce the identities

$$P_n = 2^{-\lfloor \frac{n}{2} \rfloor} \sum_{\substack{k=0 \\ 4 \nmid n-2k+1}}^{n-1} (-1)^{\lfloor \frac{(n-2k)}{4} \rfloor} \binom{2n-1}{2k+1} \text{ and } P_n = 2^{-\lfloor \frac{(n+1)}{2} \rfloor} \sum_{\substack{k=0 \\ 4 \nmid n-2k}}^{n} (-1)^{\lfloor \frac{(n-2k-1)}{4} \rfloor} \binom{2n+1}{2k+1},$$

where $P_n = F_n(2)$ is the nth Pell number.

Solution by the proposer

It is well-known that

$$F_{2n+1}(x)=\frac{1}{\sqrt{x^2+4}}\Big(\Big(\frac{x+\sqrt{x^2+4}}{2}\Big)^{2n+1}-\Big(\frac{x-\sqrt{x^2+4}}{2}\Big)^{2n+1}\Big).$$

Applying the Binomial Theorem gives

$$F_{2n+1}(x)=4^{-n}\sum_{k=0}^{n}\binom{2n+1}{2k+1}(x^2+4)^k x^{2n-2k}.$$

Replacing x by $\sqrt{ix-2}$ (here, $\sqrt{ix-2}$ can be any of the at most two possible square roots of $ix-2$) and multiplying by $(-i)^n$ yields

$$(-i)^n F_{2n+1}(\sqrt{ix-2})=4^{-n}\sum_{k=0}^{n}\binom{2n+1}{2k+1}(x-2i)^k(x+2i)^{n-k}.$$

Using the known relation $F_{2j}(y)/y=i^{1-j}F_j(i(y^2+2))$ with $y=\sqrt{ix-2}$ and noting that $F_j(-x)=(-1)^{j-1}F_j(x)$, we find

$$(-i)^n F_{2n+1}(\sqrt{ix-2})=\frac{(-i)^n}{\sqrt{ix-2}}\Big(F_{2n+2}(\sqrt{ix-2})-F_{2n}(\sqrt{ix-2})\Big)=F_{n+1}(x)+iF_n(x).$$

This proves the identity of part a.

Since $1-i=\sqrt{2}e^{-i\pi/4}$ and $1+i=\sqrt{2}e^{i\pi/4}$, the identity of **a** with $x=2$ implies that

$$P_{n+1}+iP_n=2^{-n/2}\sum_{k=0}^{n}\binom{2n+1}{2k+1}\exp\Big(\frac{(n-2k)\pi i}{4}\Big).$$

Using Euler's relation $e^{iy}=\cos y+i\sin y$, after equating the real and imaginary parts, we find

$$P_{n+1}=2^{-n/2}\sum_{k=0}^{n}\binom{2n+1}{2k+1}A_{n-2k}, \tag{1}$$

and

$$P_n=2^{-n/2}\sum_{k=0}^{n}\binom{2n+1}{2k+1}B_{n-2k}, \tag{2}$$

where $A_j = \cos(j\pi/4)$ and $B_j = \sin(j\pi/4)$, for $j \in \mathbf{Z}$. Simple calculations show that

$$A_j = \begin{cases} (-1)^{\lfloor (j+1)/4 \rfloor} 2^{\lfloor j/2 \rfloor - j/2}, & \text{if } j \not\equiv 2 \pmod 4, \\ 0, & \text{if } j \equiv 2 \pmod 4, \end{cases}$$

$$B_j = \begin{cases} (-1)^{\lfloor (j-1)/4 \rfloor} 2^{\lfloor j/2 \rfloor - j/2}, & \text{if } j \not\equiv 0 \pmod 4, \\ 0, & \text{if } j \equiv 0 \pmod 4. \end{cases}$$

Now the identities of part **b** follows from (1) with n replaced by $n-1$ and (2).

Also solved by Paul S. Bruckman.

Revisiting the Cauchy-Schwartz inequality

H-627 Proposed by Slavko Simic, Belgrade, Yugoslavia (Vol. 43, no. 3, August 2005)

Find all sequences $c = \{c_i\}_{i=1}^n$, $c_i = c_i(n)$ such that the inequality

$$|x^* - \sum_{i=1}^n c_i x_i| \le \sqrt{n-1}\sqrt{\sum_{i=1}^n c_i x_i^2 - \big(\sum_{i=1}^n c_i x_i\big)^2},$$

holds for all sequences $x = \{x_i\}_{i=1}^n$ of arbitrary real numbers and arbitrary $x^* \in x$.

Solution by the proposer

We show that the conditions of the problem are satisfied if and only if $c_i = 1/n$ for $i = 1, \dots, n$.

Putting $x_i = 1$ for all $i = 1, \dots, n$, we see that a necessary condition for the given inequality to hold is $(\sum_{i=1}^n c_i)(1 - \sum_{i=1}^n c_i) \ge 0$; i.e.,

$$0 \le \sum_{i=1}^n c_i \le 1. \tag{1}$$

Also, putting subsequently $x_i = 0$, $i \ne s$, $x_s^* = x_s = 1$, for all $s = 1, \dots, n$, in the desired inequality, we obtain

$$|1 - c_s| \le \sqrt{n-1}\sqrt{c_s(1-c_s)}, \;\; s = 1, \dots, n;$$

i.e, $c_s \geq 1/n$ for all $s = 1, \ldots, n$. But these last inequalities together with (1) imply that $c_s = 1/n$ for all $s = 1, \ldots, n$. This last condition is also sufficient since

$$(n-1)\Big(\frac{\sum_{i=1}^n x_i^2}{n} - \Big(\frac{\sum_{i=1}^n x_i}{n}\Big)^2\Big) = \frac{n-1}{n^2}\Big(n\sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2\Big)$$

$$= \frac{n-1}{n^2}\sum_{1\leq i<j\leq n}(x_i - x_j)^2 \geq \frac{1}{n^2}\Big((n-1)\sum_{i=1}^n (x_i - x^*)^2\Big) \geq \frac{1}{n^2}\big(\sum_{i=1}^n (x_i - x^*)\big)^2$$

$$= \Big(\frac{\sum_{i=1}^n x_i}{n} - x^*\Big)^2,$$

which completes the proof.

Sums of three cubes

H-628 Proposed by Juan Pla, Paris, France (Vol. 43, no. 3, August 2005)

Let us consider the set S of all the sequences $\{U_n\}_{n\geq 0}$ satisfying a second order linear recurrence

$$U_{n+2} - aU_{n+1} + bU_n = 0,$$

with both a and b rational integers, and having only integral values. Prove that for infinitely many of these sequences their general term U_n is a sum of three cubes of integers for any value of the subscript n.

Solution by the proposer

The starting point is the following easy to prove identity

$$(x+y+z)^3 - (x^3 + y^3 + z^3) = 3(x+y)(y+z)(z+x). \tag{1}$$

Setting in (1): $x = U_{n+2},\ y = -aU_{n+1},\ z = bU_n$, we obtain easily

$$U_{n+2}^3 + (-aU_{n+1})^3 + (bU_n)^3 = -3abU_{n+2}U_{n+1}U_n. \tag{2}$$

But since we have the classical relation

$$U_{n+2}U_n - U_{n+1}^2 = b^n(U_2U_0 - U_1^2),$$

after substitution of U_nU_{n+2} in the right hand side of (2) and simplifications we obtain

$$U_{n+2}^3 - (-3ab + a^3)U_{n+1}^3 + (bU_n)^3 = -3ab^{n+1}(U_2U_0 - U_1^2)U_{n+1}. \tag{3}$$

To have a sum of three cubes on the left hand side, we need only that $-3ab + b^3$ be a cube, which is the case if we set a to be an arbitrary integer and then look for b such that $-3b + a^2 = a^2c^3$, or, equivalently, $b = a^2(1-c^3)/3$, with an arbitrary integer c. In order for b to be an integer, it suffices to impose that either a is a multiple of 3 or $c \equiv 1 \pmod 3$. It is now easy to prove that the sequence whose general term appears in the right hand side of relation (3) does belong to the set S.