ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Russ Euler and Jawad Sadek

Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by November 15, 2006. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \ F_0 = 0, \ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n, \ L_0 = 2, \ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1011 (Correction) **Proposed by José Luis Díaz-Barrero, Universidad Politécnica de Cataluña, Barcelona, Spain**

Let n be a positive integer. Prove that

$$(F_n)^5 + (L_n)^5 = 2F_{n+1}\{16(F_{n+1})^4 - 20(F_{n+1})^2 F_{2n} + 5(F_{2n})^2\}.$$

B-1014 (Correction) **Proposed by Steve Edwards, Southern Polytechnic State University, Marietta, GA**

Show that $\sum_{k=0}^{\infty} \alpha^{-nk}$ equals $\frac{F_n\alpha + F_{n-1} - 1}{L_n - 2}$ if n is a positive even integer, and $\frac{F_n\alpha + F_{n-1} + 1}{L_n}$ if n is a positive odd integer.

B-1016 Proposed by Br. J. Mahon, Australia

Prove that

$$\begin{vmatrix} L_{4n+8}+1 & F_{2n+2}F_{2n+4}/3 & 1-F_{2n}^2 \\ L_{4n+4}+1 & F_{2n}F_{2n+2}/3 & 1-F_{2n-2}^2 \\ L_{4n}+1 & F_{2n-2}F_{2n}/3 & 1-F_{2n-4}^2 \end{vmatrix} = 64.$$

B-1017 Proposed by M.N. Deshpande, Nagpur, India

Define $\{a_n\}$ by $a_1 = a_2 = 0,\ a_3 = a_4 = 1$ and

$$a_n = a_{n-1} + a_{n-3} + a_{n-4} + k(n)$$

for $n \geq 5$ where

$$k(n) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ i^{n-2} & \text{if } n \text{ is odd} \end{cases}$$

and $i\ \sqrt{-1}$.

Prove or disprove: $a_n + 2a_{n+2} + a_{n+4}$ is a Fibonacci number for all integers $n \geq 1$.

B-1018 Proposed by Mohammad K. Azarian, Evansville, Indiana

If $n \geq 2$ and $\binom{n}{k}$ denotes the binomial coefficient, show that

$$\begin{aligned} f(F,L) = & F_{n+2} + (F_2 + F_4 + F_6 + \cdots + F_{2n} - F_{2n+1})^{F_{n+1}} \binom{L_n}{F_{n+1}} \binom{L_n}{F_{n-1}} \binom{-F_{n-1}-1}{F_{n+1}} \\ & + (F_2 + F_4 + F_6 + \cdots + F_{2n} - F_{2n+1})^{1+F_{n+1}} \binom{L_n}{F_{n+1}} \binom{-F_{n-1}-1}{F_{n+1}} \\ & + (F_2 + F_4 + F_6 + \cdots + F_{2n} - F_{2n+1})^{1+F_{n+1}} \binom{-F_{n-1}-1}{F_{n+1}} \\ & - (F_1 + F_2 + F_3 + \cdots + F_n) \end{aligned}$$

can be written in the form MN^2, where M and N are natural numbers.

SOLUTIONS

Inverse sines, Fibonacci, and Pi!

B-1001 Proposed by Paul S. Bruckman, Canada
(Vol. 43, no. 3, August 2005)

(a) Let $\theta(a,b) = \sin^{-1}\{(a^2-b^2)/(a^2+b^2)\}$ denote one of the acute angles of a Pythagorean triangle, where a and b are integers with $a > b > 0$. Given $\theta(a,b)$ and $\theta(c,d)$, show that there exists $\theta(e,f)$, where e and f are functions of a, b, c and d, such that

$$\theta(a,b) + \theta(c,d) + \theta(e,f) = \pi/2.$$

(b) As a special case, prove the following identity:

$$\sin^{-1}\{F_{n-1}F_{n+2}/F_{2n+1}\} + \sin^{-1}\{L_{n-1}L_{n+2}/5F_{2n+1}\} + \sin^{-1}\{3/5\} = \pi/2.$$

Solution by Jaroslav Seibert, University Hradec Kralove, The Czech Republic

(a) First, we write the given equality in the form

$$\theta(a,b) + \theta(e,f) = \frac{\pi}{2} - \theta(c,d). \tag{1}$$

Since $\theta(c,d)$ denotes one of the acute angles of a Pythagorean triangle, $\frac{\pi}{2} - \theta(c,d)$ denotes the second of these acute angles. Because the triangle is a Pythagorean tirangle, the lengths of its sides may be written as $c^2+d^2,\ c^2-d^2,\ 2cd$, and the relation $\frac{\pi}{2} - \theta(c,d) = \sin^{-1}\frac{2cd}{c^2+d^2}$ holds.

We will consider only such acute angles $\theta(a,b)$ and $\theta(c,d)$ whose sum is less than $\frac{\pi}{2}$. It is a well-known formula (see [1], p. 111) that

$$\sin^{-1}x - \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} - y\sqrt{1-x^2}) \text{ if } x^2+y^2 \le 1. \tag{2}$$

We can easily show that the last condition is valid if $0 < x+y \le \frac{\pi}{2}$.

Equality (1) can be rewritten in the form

$$\sin^{-1}\frac{e^2-f^2}{e^2+f^2} = \sin^{-1}\frac{2cd}{c^2+d^2} - \sin^{-1}\frac{a^2-b^2}{a^2+b^2}.$$

Using (2) we get

$$\begin{aligned}\sin^{-1}\frac{e^2-f^2}{e^2+f^2} &= \sin^{-1}\left(\frac{2cd}{c^2+d^2}\sqrt{1-\left(\frac{a^2-b^2}{a^2+b^2}\right)^2} - \frac{a^2-b^2}{a^2+b^2}\sqrt{1-\left(\frac{2cd}{c^2+d^2}\right)^2}\right)\\ &= \sin^{-1}\frac{4abcd-(a^2-b^2)(c^2-d^2)}{(a^2+b^2)(c^2+d^2)}.\end{aligned}$$

The last equality follows if we can show

$$e^2 - f^2 = 4abcd - (a^2-b^2)(c^2-d^2) \tag{3}$$

and

$$e^2 + f^2 = (a^2+b^2)(c^2+d^2). \tag{4}$$

To prove (3) we write

$$\begin{aligned}e^2 - f^2 &= 4abcd - (a^2c^2 - a^2d^2 - b^2c^2 + b^2d^2)\\ &= (a^2d^2 + 2abcd + b^2c^2) - (a^2c^2 - 2abcd + b^2d^2) = (ad+bc)^2 - (ac-bd)^2.\end{aligned} \tag{5}$$

Therefore we can put $e = ad + bc$ and $f = ac - bd$.

It follows easily that equality (4) is also true. In fact,

$$\begin{aligned}e^2 + f^2 = (ad+bc)^2 + (ac-bd)^2 &= a^2d^2 + 2abcd + b^2c^2 + a^2c^2 - 2abcd + b^2d^2\\ &= (a^2+b^2)(c^2+d^2).\end{aligned}$$

(b) Using the following identities ([2], identities (14), (16a), (25)),

$$F_{n+1}L_{n+1} - F_nL_n = F_{2n+1},\ F_nL_{n+1} + F_{n+1}L_n = 2F_{2n+1},$$

$$\text{and } 5(F_n^2 + F_{n+1}^2) = L_n^2 + L_{n+1}^2 = 5F_{2n+1},$$

we can write

$$\frac{F_{n-1}F_{n+2}}{F_{2n+1}} = \frac{(F_{n+1} - F_n)(F_{n+1} + F_n)}{F_{n+1}^2 + F_n^2} = \frac{F_{n+1}^2 - F_n^2}{F_{n+1}^2 + F_n^2},$$

and

$$\frac{L_{n-1}L_{n+2}}{5F_{2n+1}} = \frac{L_{n+1}^2 - L_n^2}{L_{n+1}^2 + L_n^2}.$$

Putting $a = F_{n+1}, b = F_n, c = L_{n+1}, d = L_n$ in relations (5), we get

$$e = F_{n+1}L_n + F_nL_{n+1} = 2F_{2n+1}, f = F_{n+1}L_{n+1} - F_nL_n = F_{2n+1},$$

and

$$\frac{e^2 - f^2}{e^2 + f^2} = \frac{4F_{2n+1}^2 - F_{2n+1}^2}{4F_{2n+1}^2 + F_{2n+1}^2} = \frac{3}{5}$$

which completes the proof of (b).

References.

1. K. Rektorys. Prehled uzite matematiky, SNTL, Praha 1973.
2. S. Vajda. Fibonacci and Lucas Numbers, and the Golden Section, Chichester: Ellis Horwood Ltd, 1989.

Also solved by Russell J. Hendel and the proposer.

A Series Identity

B-1002 Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 43, no. 3, August 2005)

Prove the identity

$$\sum_{k=1}^{\infty}(-1)^k\left(\frac{F_{k+1}}{F_k} - \alpha\right) = \frac{1}{2}\left(-\beta + \sum_{k=1}^{\infty}\frac{1}{F_kF_{k+1}}\right).$$

Solution by Russell Jay Hendel, Towson University, Towson, MD

[I] I_{102} (identity 102) states

$$-\beta = \sum_{i=2}^{\infty}\frac{(-1)^n}{F_{n-1}F_n} = \sum_{i=1}^{\infty}\frac{1}{F_{2n-1}F_{2n}} - \sum_{i=1}^{\infty}\frac{1}{F_{2n}F_{2n+1}}. \tag{1}$$

(The rearrangement is justified because the series is majorized by a convergent geometric series.) To relate this to the problem sums note that by [1], I_{29},

$$\left(\frac{F_{2n+1}}{F_{2n}} - \alpha\right) + \left(\alpha - \frac{F_{2n}}{F_{2n-1}}\right) = \frac{F_{2n+1}}{F_{2n}} - \frac{F_{2n}}{F_{2n-1}} = \frac{1}{F_{2n-1}F_{2n+1}},$$

implying that

$$\sum_{k=1}^{\infty}(-1)^k\left(\frac{F_{k+1}}{F_k} - \alpha\right) = \sum_{k=1}^{\infty}\frac{1}{F_{2k-1}F_{2k}}. \tag{2}$$

It immediately follows that

$$\begin{aligned}
2\sum_{k=1}^{\infty}&(-1)^k\left(\frac{F_{k+1}}{F_k} - \alpha\right) - \sum_{k=2}^{\infty}\frac{1}{F_{k-1}F_k} \\
&= 2\sum_{k=1}^{\infty}\frac{1}{F_{2k-1}F_{2k}} - \sum_{k=2}^{\infty}\frac{1}{F_{k-1}F_k}, \text{ by (2)}, \\
&= \sum_{k=1}^{\infty}\frac{1}{F_{2k-1}F_{2k}} - \sum_{k=1}^{\infty}\frac{1}{F_{2k}F_{2k+1}}, \quad \text{by series algebra}, \\
&= -\beta, \quad \text{by (1)}.
\end{aligned}$$

The problem identity immediately follows.

References.

1. S. Vajda. *Fiboancci and Lucas Numbers and the Golden Section Theory and Applications.* John Wiley, 1989, pp. 177-184.

Also solved by Paul S. Bruckman, Jaroslav Seibert, and the proposer.

The Index is Odd, The Number is Squared

B-1003 Proposed by Paul S. Bruckman, Canada (Vol. 43, no. 3, August 2005)

Prove the identity

$$F_{n+2}L_{n+1}L_nF_{n-1} + L_{n+2}F_{n+1}F_nL_{n-1} = 2(F_{2n+1})^2.$$

Solution by H.-J. Seiffert, Berlin, Germany

¿From eqns. (3.22) - (3.25) of [1], we have

$$\begin{aligned}
5F_{n+2}F_{n-1} &= L_{2n+1} + 4(-1)^n, \quad L_{n+1}L_n = L_{2n+1} + (-1)^n, \\
L_{n+2}L_{n-1} &= L_{2n+1} - 4(-1)^n, \quad 5F_{n+1}F_n = L_{2n+1} - (-1)^n, \\
L_{2n+1}^2 &= L_{4n+2} - 2, \quad \text{and} \quad 5F_{2n+1}^2 = L_{4n+2} + 2.
\end{aligned}$$

Hence, if A_n denotes the expression on the left hand side of the desired identity, then

$$\begin{aligned}5A_n &= (L_{2n+1} + 4(-1)^n)(L_{2n+1} + (-1)^n) \\ &\quad + (L_{2n+1} - 4(-1)^n)(L_{2n+1} - (-1)^n) \\ &= 2(L_{2n+1}^2 + 4) = 2(L_{4n+2} + 2) = 10F_{2n+1}^2,\end{aligned}$$

giving $A_n = 2F_{2n+1}^2$.

References.

1. A.F. Horadam and Bro. J.M. Mahon. "Pell and Pell-Lucas Polynomials." *The Fibonacci Quarterly*, **23.1** (1985): 7-20.

Also solved by Brian D. Beasley, G.C. Greubel, Ralph P. Grimaldi, Russell J. Hendel, Kathleen E. Lewis, David E. Maner, Jaroslav Seibert, and the proposer.

Double the Trouble!

B-1004 Proposed by José Luis Díaz-Barrero, Barcelona, Spain (Vol. 43, no. 3, August 2005)

Let n be a positive integer. Calculate

$$\lim_{n\to\infty}\left\{\frac{1}{F_n}\left(\frac{\sum_{i=1}^n\sum_{k=1}^n F_{|i+k|}}{\sum_{i=1}^n\sum_{k=1}^n F_{|i-k|}}\right)\right\}.$$

Solution by Steve Edwards, Southern Polytechnic State University, Marietta, GA

We use the formula $\sum_{k=j}^n F_k = F_{n+2} - F_{j+1}$ which can easily be shown by induction

$$\sum_{i=1}^n\sum_{k=1}^n F_{|i+k|} = \sum_{i=1}^n F_{n+i+2} - F_{i+2} = F_{2n+4} - F_{n+4} - (F_{n+4} - F_4) = F_{2n+4} - 2F_{n+4} - F_4.$$

Next note that by symmetry and the fact that $F_0 = 0$, $\sum_{i=1}^n\sum_{k=1}^n F_{|i-k|} = 2\sum_{i=1}^{n-1}\sum_{j=1}^i F_j$.

Using the formula above, this becomes $2\sum_{i=1}^{n-1} F_{i+2} - F_2 = 2[F_{n+3} - F_4 - (n-1)] = 2[F_{n+3} - n - 4]$. This means that $\frac{1}{F_n}\left(\frac{\sum_{i=1}^n\sum_{k=1}^n F_{|i+k|}}{\sum_{i=1}^n\sum_{k=1}^n F_{|i-k|}}\right) = \frac{F_{2n+4}-2F_{n+4}-F_4}{2F_n(F_{n+3}-n-4)}$. Now since $F_k = \frac{\alpha^k-\beta^k}{\sqrt{5}}$, while $\lim_{n\to\infty}\beta^n = 0$, we have $\lim_{n\to\infty}\left\{\frac{1}{F_n}\left(\frac{\sum_{i=1}^n\sum_{k=1}^n F_{|i+k|}}{\sum_{i=1}^n\sum_{k=1}^n F_{|i-k|}}\right)\right\} = \frac{\sqrt{5}\alpha}{2}$, which is also equivalent to $\frac{\alpha^2+1}{2}$ or $\frac{\alpha}{2}+1$.