ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
FLORIAN LUCA

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, IMATE, UNAM, AP. POSTAL 61-3 (XANGARI), CP 58 089, MORELIA, MICHOACAN, MEXICO, or by e-mail at fluca@matmor.unam.mx as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-700 Proposed by Mohamed El Bachraoui, United Arab Emirates

Let μ be the Möbius mu function and let $M(n)$ be the Mertens function given by $M(n) = \sum_{a \le n} \mu(a)$. If $n > 2$, it is clear that

$$M(n) \equiv \#\{a \in [2, n-1] : \ a \text{ squarefree}, \ a \nmid n\} \pmod 2.$$

Prove that for all positive integers $n > 2$ we have

a) $M(2n) \equiv 1 + \#\{a \in [2, 2n-3] : \ a \text{ squarefree}, \ a \nmid 2n, \ a \nmid 2n-1, \ a \nmid 2n-2\} \pmod 2$;
b) $M(2n+1) \equiv \#\{a \in [2, 2n-2] : \ a \text{ squarefree}, \ a \nmid 2n+1, \ a \nmid 2n, a \nmid 2n-1\} \pmod 2$.

H-701 Proposed by Sergio Falcón and Ángel Plaza, Gran Canaria, Spain

For $k \ge 1$, let $F_{k,n}$ be the sequence given by $F_{k,0} = 0, \ F_{k,1} = 1, \ F_{k,n+2} = kF_{k,n+1} + F_{k,n}$ for $n \ge 0$. Show that if $2r + h \ne 0$, then

$$\frac{F_{k,n+r}F_{k,n+r+h} + (-1)^{h+1}F_{k,n-r}F_{k,n-r-h}}{F_{k,2r+h}} = F_{k,2n}.$$

H-702 Proposed by Hideyuki Ohtsuka, Saitama, Japan

For an integer $m \ne 0$ determine

$$\sum_{k=1}^{\infty} \frac{4^k}{L_{m2^k}^2}.$$

H-703 Proposed by Napoleon Gauthier, Kingston, ON

Let n be a positive integer and prove the following identities:

a) $\displaystyle\sum_{k \ge 0} k\binom{n-k-1}{k} = \frac{1}{10}[(5n-4)F_n - nL_n]$;

b) $\displaystyle\sum_{k \ge 0} k^2\binom{n-k-1}{k} = \frac{1}{50}[(15n^2 - 20n + 4)F_n - (5n^2 - 6n)L_n]$.

SOLUTIONS

Laguerre Meets Fibonacci

H-682 Proposed by G. C. Greubel, Newport News, VA (Vol. 47, No. 1, February 2009/2010)

Given the generalized Laguerre polynomials,

$$L_n^{(\alpha,\beta,\gamma)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_2F_2(-n,\beta+1;\alpha+1,\gamma+1;x),$$

show that

$$L_n^{(\alpha,\beta,\gamma)}\left(x^2\right) = \frac{(\alpha+1)_n}{n!}\sum_{r=0}^{n}\frac{(-n)_r(\beta+1)_r}{r!(\alpha+1)_r(\gamma+1)_r}\cdot\theta_r^n\cdot F_{2r+1}(x),$$

where $F_n(x)$ is the Fibonacci polynomial and θ_r^n is a hypergeometric polynomial of type ${}_3F_3$.

Solution by the proposer

It can be seen that

$$\frac{x^n}{n!} = \sum_{k=0}^{[n/2]}\frac{(-1)^k(n-2k+1)}{k!(n-k+1)!}\cdot F_{n-2k+1}(x) \tag{1}$$

which will be required later. Consider the series

$$\sum_{n=0}^{\infty} L_n^{(\alpha,\beta,\gamma)}\left(x^2\right)\frac{t^n}{(\alpha+1)_n}$$

which will be used to show the connection between the generalized Laguerre polynomials and the Fibonacci polynomials.

$$\begin{aligned}S &= \sum_{n=0}^{\infty} L_n^{(\alpha,\beta,\gamma)}\left(x^2\right)\frac{t^n}{(\alpha+1)_n} = \sum_{n=0}^{\infty}\sum_{r=0}^{n}\frac{(-n)_r(\beta+1)_r}{(\alpha+1)_r(\gamma+1)_r}\cdot\frac{x^{2r}t^n}{n!r!}\\ &= \sum_{n,r=0}^{\infty}\frac{(-n-r)_r(\beta+1)_r x^{2r}t^{n+r}}{(n+r)!r!(\alpha+1)_r(\beta+1)_r} = \sum_{n,r=0}^{\infty}\frac{(-1)^r(\beta+1)_r x^{2r}t^{n+r}}{n!r!(\alpha+1)_r(\beta+1)_r}\\ &= \sum_{n,r=0}^{\infty}\frac{(-1)^r(\beta+1)_r(1)_{2r}t^{n+r}}{n!r!(\alpha+1)_r(\beta+1)_r}\cdot\frac{x^{2r}}{(2r)!}.\end{aligned}$$

By using equation (1) we have

$$\begin{aligned}S &= \sum_{n,r=0}^{\infty}\sum_{s=0}^{r}\frac{(-1)^{r+s}(1)_{2r}(2r-2s+1)(\beta+1)_r t^{n+r}}{r!n!s!(2r-s+1)!(\alpha+1)_r(\gamma+1)_r}F_{2r-2s+1}(x)\\ &= \sum_{n,r,s=0}^{\infty}\frac{(-1)^r(1)_{2r+2s}(2r+1)(\beta+1)_{r+s}t^{n+r+s}}{(r+s)!n!s!(2r+s+1)!(\alpha+1)_{r+s}(\gamma+1)_{r+s}}F_{2r+1}(x)\\ &= \sum_{n,r=0}^{\infty}\sum_{s=0}^{n}\frac{(-1)^r(2r+1)(2r+2s)!(\beta+1)_{r+s}}{(n-s)!(2r+s+1)!(\alpha+1)_{r+s}(\gamma+1)_{r+s}}\frac{t^{n+r}}{s!(r+s)!}F_{2r+1}(x).\end{aligned}$$

Thus,

$$S = \sum_{n,r=0}^{\infty} \frac{(-1)^r(\beta+1)_r t^{n+r}}{n!r!(\alpha+1)_r(\gamma+1)_r} F_{2r+1}(x) \times \sum_{s=0}^{n} \frac{(-1)^s(-n)_s(\beta+r+1)_s(2r+1)_{2s}}{s!(r+1)_s(2r+2)_s(\alpha+r+1)_s(\gamma+r+1)_s}. \tag{2}$$

Let ϕ_r^n be given by

$$\phi_r^n = \sum_{s=0}^{n} \frac{(-1)^s(-n)_s(\beta+r+1)_s(2r+1)_{2s}}{s!(r+1)_s(2r+2)_s(\alpha+r+1)_s(\gamma+r+1)_s}.$$

A reduction of this series is seen as follows:

$$\begin{aligned}\phi_r^n &= \sum_{s=0}^{n} \frac{(-1)^s(-n)_s(\beta+r+1)_s(2r+1)_{2s}}{s!(r+1)_s(2r+2)_s(\alpha+r+1)_s(\gamma+r+1)_s} \\ &= \sum_{s=0}^{n} \frac{(-4)^s(-n)_s(\beta+r+1)_s\left(r+\frac{1}{2}\right)_s}{s!(2r+2)_s(\alpha+r+1)_s(\gamma+r+1)_s} \\ &= {}_3F_3\left(-n, r+\frac{1}{2}, \beta+r+1; 2r+2; \alpha+r+1, \gamma+r+1; -4\right).\end{aligned} \tag{3}$$

With equation (3) in mind we have from equation (2)

$$\begin{aligned}S &= \sum_{n,r=0}^{\infty} \frac{(-1)^r(\beta+1)_r\phi_r^n t^{n+r}}{n!r!(\alpha+1)_r(\gamma+1)_r} F_{2r+1}(x) \\ &= \sum_{n=0}^{\infty}\sum_{r=0}^{n} \frac{(-1)^r(\beta+1)_r\phi_r^{n-r} t^{n}}{(n-r)!r!(\alpha+1)_r(\gamma+1)_r} F_{2r+1}(x).\end{aligned}$$

From this we have

$$L_n^{(\alpha,\beta,\gamma)}\left(x^2\right) = \frac{(\alpha+1)_n}{n!}\sum_{r=0}^{n} \frac{(-n)_r(\beta+1)_r}{r!(\alpha+1)_r(\gamma+1)_r} \cdot \theta_r^n F_{2r+1}(x), \tag{4}$$

where $\theta_r^n = \phi_r^{n-r}$ is given by

$$\theta_r^n = {}_3F_3\left(-n+r, r+\frac{1}{2}, \beta+r+1; 2r+2; \alpha+r+1, \gamma+r+1; -4\right). \tag{5}$$

Also solved by Paul S. Bruckman.

Identities With Symmetric Polynomials

H-683 Proposed by Guodong Liu, Huizhou, China
(Vol. 47, No. 1, February 2009/2010)

The generalized binomial coefficients of the first kind $\sigma_k(x_1, x_2, \ldots, x_n)$ and the generalized binomial coefficients of the second kind $\tau_k(x_1, x_2, \ldots, x_n)$ are defined by

$$(1 - x_1 x)(1 - x_2 x) \cdots (1 - x_n x) = \sum_{k=0}^{\infty} \sigma_k(x_1, x_2, \ldots, x_n) x^k$$

and

$$\frac{1}{(1 - x_1 x)(1 - x_2 x) \cdots (1 - x_n x)} = \sum_{k=0}^{\infty} \tau_k(x_1, x_2, \ldots, x_n) x^k,$$

respectively. For any positive integers n and k prove that

$$x_1^k + x_2^k + \cdots + x_n^k = -\sum_{j=1}^{k} j \sigma_j(x_1, \ldots, x_n) \tau_{k-j}(x_1, \ldots, x_n).$$

Solution by Eduardo H. M. Brietzke, UFRGS, Brasil

The first series is a polynomial, as $\sigma_k = 0$ for $k > n$, whereas the second power series has a positive radius of convergence equal to $\min\{|x_1|^{-1}, \ldots, |x_n|^{-1}\}$, since the left-hand-side of it is a product of n geometric series

$$\frac{1}{1 - x_i x} = \sum_{k=0}^{\infty} x_i^k x^k, \qquad (|x| < |x_i|^{-1}).$$

Differentiating the first relation in the statement of the problem and multiplying the answer by x, we obtain

$$-\sum_{i=1}^{n} \frac{x_i x (1 - x_1 x)(1 - x_2 x) \cdots (1 - x_n x)}{1 - x_i x} = \sum_{k=1}^{\infty} k \sigma_k(x_1, x_2, \ldots, x_n) x^k. \tag{6}$$

Multiplying the second relation in the statement of the problem by (6), yields

$$-\sum_{i=1}^{n} \frac{x_i x}{1 - x_i x} = \sum_{k=1}^{\infty} x^k \sum_{j=1}^{k} j \sigma_j(x_1, x_2, \ldots, x_n) \tau_{k-j}(x_1, x_2, \ldots, x_n).$$

Hence,

$$-\sum_{i=1}^{n} \sum_{k=1}^{\infty} x_i^k x^k = \sum_{k=1}^{\infty} x^k \sum_{j=1}^{k} j \sigma_j(x_1, x_2, \ldots, x_n) \tau_{k-j}(x_1, x_2, \ldots, x_n). \tag{7}$$

Comparing the coefficients of x^k in both sides of (7) yields immediately the desired conclusion.

Also solved by Paul S. Bruckman and Harris Kwong.

Fugue on a Generalized Fibonacci Sequence Theme

H-684 Proposed by N. Gauthier, Kingston, ON (Vol. 47, No. 1, February 2009/2010)

For an arbitrary positive integer N and a real number $a > 2$, consider the following $N \times N$ matrix:

$$\mathbf{A} = \begin{pmatrix} a & 1 & 0 & \dots & 0 & 0 \\ 1 & a & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & a & 1 \\ 0 & 0 & 0 & \dots & 1 & a \end{pmatrix}.$$

(a) Find a closed form expression for the determinant of A.
(b) Find the eigenvalues of A and show that they are real, positive, and distinct.
(c) Find expressions for the eigenvectors of $\mathbf{A}$.

Solution by Ángel Plaza and Sergio Falcón, Gran Canaria, Spain

Let us denote by $\mathbf{A}_n$ the given $n \times n$ matrix and for a square matrix $\mathbf{A}$ we put $|\mathbf{A}|$ for its determinant.

(a) By expanding on the first column, we get $|\mathbf{A}_n| = a\,|\mathbf{A}_{n-1}| - |\mathbf{A}_{n-2}|$. Since $|\mathbf{A}_1| = a$ and $|\mathbf{A}_2| = a^2 - 1$, solving the associated recurrence equation we obtain

$$|\mathbf{A}_n| = \frac{\alpha^{n+1} - \beta^{n+1}}{\sqrt{a^2 - 4}},$$

where $\alpha = \dfrac{a + \sqrt{a^2 - 4}}{2}$ and $\beta = \dfrac{a - \sqrt{a^2 - 4}}{2}$ are the roots of the characteristic equation $x^2 = ax - 1$. Note that the sequence $\{|\mathbf{A}_n|\}_{n \geq 1}$ is a generalized Fibonacci sequence. □

(b) Taking into account that the Chebyshev polynomials of the second kind, $U_n(x)$, obey the interesting identity

$$U_n(x) = \begin{vmatrix} 2x & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 2x & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 2x & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2x & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 2x & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 2x \end{vmatrix},$$

and since the roots of $U_n(x)$ are $x_k = \cos\left(\frac{k}{n+1}\pi\right)$ for $k = 1, \dots, n$, we have that the eigenvalues of $\mathbf{A}$ are $\lambda_n = a + 2\cos\left(\frac{k\pi}{n+1}\right)$ for $k = 1, \dots, n$. Since $a > 2$, all the eigenvalues are real, positive, and distinct. □

(c) To find the eigenvectors of $\mathbf{A_n}$ we have to solve $(\mathbf{A_n} - \lambda \mathbf{I_n})\mathbf{x} = \mathbf{0}$.

For each $k = 1, \ldots, n$ the corresponding eigenvalue is $\lambda = a + 2\cos\left(\frac{k\pi}{n+1}\right)$ and the system of equations to be solved reads as

$$\begin{pmatrix} -2\cos\left(\frac{k\pi}{n+1}\right) & 1 & \cdots & 0 & 0 \\ 1 & -2\cos\left(\frac{k\pi}{n+1}\right) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -2\cos\left(\frac{k\pi}{n+1}\right) & 1 \\ 0 & 0 & \cdots & 1 & -2\cos\left(\frac{k\pi}{n+1}\right) \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \mathbf{0}$$

That is,

$$-2\cos\left(\frac{k\pi}{n+1}\right) x_1 + x_2 = 0,$$
$$x_{m-1} - 2\cos\left(\frac{k\pi}{n+1}\right) x_m + x_{m+1} = 0 \quad \text{for} \quad m = 2, \ldots, n-1, \quad \text{and}$$
$$x_{n-1} - 2\cos\left(\frac{k\pi}{n+1}\right) x_n = 0.$$

Let us take $x_1 = 1$. Solving the associated recurrence equation we get, for $m = 1, \ldots, n$:

$$x_m = \frac{\alpha^m - \beta^m}{2i\sin\left(\frac{k\pi}{n+1}\right)},$$

where

$$\alpha = \cos\left(\frac{k\pi}{n+1}\right) + i\sin\left(\frac{k\pi}{n+1}\right) = e^{\frac{k\pi i}{n+1}}$$
$$\beta = \cos\left(\frac{k\pi}{n+1}\right) - i\sin\left(\frac{k\pi}{n+1}\right) = e^{-\frac{k\pi i}{n+1}}$$

are the roots of the characteristic equation $x^2 - 2\cos\left(\frac{k\pi}{n+1}\right) x + 1 = 0$. After some algebra, we get

$$x_m = \frac{\sin\left(\frac{mk\pi}{n+1}\right)}{\sin\left(\frac{k\pi}{n+1}\right)}, \quad \text{for} \quad m = 1, 2, \ldots, n.$$

Thus, the eigenvector associated to $\lambda = a + 2\cos\left(\frac{k\pi}{n+1}\right)$ may be written as

$$\mathbf{x} = \left(\sin\left(\frac{k\pi}{n+1}\right), \sin\left(\frac{2k\pi}{n+1}\right), \ldots, \sin\left(\frac{nk\pi}{n+1}\right)\right).$$

Also solved by Michael R. Bacon and Charles K. Cook (jointly), Paul S. Bruckman, Harris Kwong and the proposer.