ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Florian Luca

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, IMATE, UNAM, AP. POSTAL 61-3 (XANGARI), CP 58 089, MORELIA, MICHOACAN, MEXICO, or by e-mail at fluca@matmor.unam.mx as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within six months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-647 Proposed by N. Gauthier, Kingston, ON

For a positive integer n and a non-zero number k, consider the following recurrence for generalized Fibonacci numbers

$$f_{r+2} = kf_{r+1} + f_r; \qquad r \geq 0; \qquad f_0 = 0, \ f_1 = 1.$$

a). Prove the following two identities

$$\sum_{r=0}^{n-1} rf_r = \frac{n}{k}(f_n + f_{n-1}) + \frac{1}{k^2}(2 - (k+2)f_n - 2f_{n-1});$$

$$\sum_{r=0}^{n-1} r^2 f_r = \frac{n^2}{k}(f_n+f_{n-1})+\frac{1}{k^2}(2-(2n+1)((k+2)f_n+2f_{n-1}))-\frac{2}{k^3}((k+4)-(k^2+3k+4)f_n-(k+4)f_{n-1}).$$

b). Find a similar formula for $\sum_{r=0}^{n-1} r^3 f_r$, in terms of f_n, f_{n-1} and a constant term.

H-648 Proposed by Ovidiu Furdui, Kalamazoo, MI

Let n be a positive integer. Prove that the following identity holds

$$L_{n+1} = (n+1)\left(\sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \frac{1}{\lfloor \frac{n-1}{2} \rfloor - j + 1} \cdot \binom{\lfloor \frac{n}{2} \rfloor + j}{\lfloor \frac{n-1}{2} \rfloor - j}\right) + 1.$$

H-649 Proposed by Stanley Rabinowitz, Chelmsford, MA

Find positive integers a, b and c such that $\sec(F_a) + \sec(F_b) = \sec(F_c)$, where all angles are measured in degrees.

SOLUTIONS

A wicked identity

H-614 Proposed by R.S. Melham, Sydney, Australia (Vol. 42, no. 3, August 2004)

Prove the identity

$$F_{a_2-a_3}F_{a_2-a_4}F_{a_3-a_4}F^2_{n+a_1}+(-1)^{a_1+a_2+1}F_{a_1-a_3}F_{a_1-a_4}F_{a_3-a_4}F^4_{n+a_2}$$

$$+(-1)^{a_1+a_2}F_{a_1-a_2}F_{a_1-a_4}F_{a_2-a_4}F^2_{n+a_3}+(-1)^{a_1+a_2+a_3+a_4+1}F_{a_1-a_2}F_{a_1-a_3}F_{a_2-a_3}F^4_{n+a_4}$$
$$=F_{a_1-a_2}F_{a_1-a_3}F_{a_1-a_4}F_{a_2-a_3}F_{a_2-a_4}F_{a_3-a_4}F_{4n+a_1+a_2+a_3+a_4}.$$

Editor's solution based on a solution by Kenny Davenport

We use $F_n=(\alpha^n-\beta^n)/(\alpha-\beta)$ for all $n\in\mathbb{Z}$, where $\alpha=(1+\sqrt{5})/2,\ \beta=(1-\sqrt{5})/2$, and $\alpha\beta=-1$.

Lemma. *The identity*

$$F_xF_yF_{-x-y}=\frac{1}{(\alpha-\beta)^2}((-1)^xF_{2x}+(-1)^yF_{2y}-(-1)^{x+y}F_{2x+2y})$$

holds for all integers $x,\ y$.

Proof of the Lemma: The desired identity follows since

$$F_xF_yF_{-x-y}=\frac{1}{(\alpha-\beta)^3}(\alpha^x-(-\alpha)^{-x})(\alpha^y-(-\alpha)^{-y})(\alpha^{-x-y}-(-\alpha)^{x+y})$$

$$=\frac{1}{(\alpha-\beta)^3}(\alpha^{2x}-(-1)^x)(\alpha^{2y}-(-1)^y)(\alpha^{-2x-2y}-(-1)^{x+y})$$

$$=\frac{1}{(\alpha-\beta)^3}(\alpha^{2x+2y}-(-1)^y\alpha^{2x}-(-1)^x\alpha^{2y}+(-1)^{x+y})(\alpha^{-2x-2y}-(-1)^{x+y})$$

$$=\frac{1}{(\alpha-\beta)^3}(1-(-1)^{x+y}(\alpha^{2x+2y}-\alpha^{-2x-2y})+(-1)^x(\alpha^{2x}-\alpha^{-2x})+(-1)^y(\alpha^{2y}-\alpha^{-2y})-1)$$

$$=\frac{1}{5}((-1)^xF_{2x}+(-1)^yF_{2y}-(-1)^{x+y}F_{2x+2y}).$$

Write

$$F_{n+a}^4=\left(\frac{\alpha^{n+a}-\beta^{n+a}}{\alpha-\beta}\right)^4=\frac{1}{25}\left(\alpha^{4a+4n}+\beta^{4a+4n}-\binom{4}{1}(-1)^{n+a}(\alpha^{2n+2a}+\beta^{2n+2a})+\binom{4}{2}\right), \tag{1}$$

with $a=a_i$ for $i=1,2,3,4$. Using the Binet formula for $F_{4n+a_1+a_2+a_3+a_4}$, we see easily that it suffices to prove that the following three identities hold

$$\alpha^{4a_1}F_{a_2-a_3}F_{a_2-a_4}F_{a_3-a_4}+(-1)^{a_1+a_2+1}\alpha^{4a_2}F_{a_1-a_3}F_{a_1-a_4}F_{a_3-a_4}$$
$$+(-1)^{a_1+a_2}\alpha^{4a_3}F_{a_1-a_2}F_{a_1-a_4}F_{a_2-a_4}+(-1)^{a_1+a_2+a_3+a_4+1}\alpha^{4a_4}F_{a_1-a_2}F_{a_1-a_3}F_{a_2-a_3}$$
$$=5\sqrt{5}F_{a_1-a_2}F_{a_1-a_3}F_{a_1-a_4}F_{a_2-a_3}F_{a_2-a_4}F_{a_3-a_4}\alpha^{a_1+a_2+a_3+a_4}; \tag{2}$$

$$(-1)^{a_1}\alpha^{2a_1}F_{a_2-a_3}F_{a_2-a_4}F_{a_3-a_4}+(-1)^{a_1+1}\alpha^{2a_2}F_{a_1-a_3}F_{a_1-a_4}F_{a_3-a_4}$$
$$+(-1)^{a_1+a_2+a_3}\alpha^{a_3}F_{a_1-a_2}F_{a_1-a_4}F_{a_2-a_4}+(-1)^{a_1+a_2+a_3+1}\alpha^{a_4}F_{a_1-a_2}F_{a_1-a_3}F_{a_2-a_3}=0, \tag{3}$$

and

$$F_{a_2-a_3}F_{a_2-a_4}F_{a_3-a_4}+(-1)^{a_1+a_2+1}F_{a_1-a_3}F_{a_1-a_4}F_{a_3-a_4}+(-1)^{a_1+a_2}F_{a_1-a_2}F_{a_1-a_4}F_{a_2-a_4}$$
$$+(-1)^{a_1+a_2+a_3+a_4+1}F_{a_1-a_2}F_{a_1-a_3}F_{a_2-a_3}=0. \tag{4}$$

Indeed, note that if relations (2) and (3) are satisfied, then the relations obtained from (2) and (3) by conjugation (i.e., by replacing $\sqrt{5}$ by $-\sqrt{5}$; hence, in particular, α by β) are also satisfied, and now Binet's formulas together with formula (1) for $a=a_i$, $i=1,2,3,4$ imply the desired conclusion.

In what follows, we shall concentrate on proving formulas (2), (3) and (4). The method consists in expanding both sides of the above relations as linear combinations of powers of α with certain coefficients and showing that the same powers with the same coefficients show up in both sides. Given a fixed vector $\boldsymbol{c}=(c_1,c_2,c_3,c_4)$ and a permutation σ of $\{1,2,3,4\}$, we need to justify that $\alpha^{c_1a_{\sigma(1)}+c_2a_{\sigma(2)}+c_3a_{\sigma(3)}+c_4a_{\sigma(4)}}$ appears with the same coefficients in each of the two sides of the corresponding identity to be proved. In general, we shall justify only the case of the identical permutation σ, since the remaining ones can be dealt with similarly.

We start with (4). To make some sense of the signs, note that each term involves a product of three Fibonacci numbers in the three variables $\boldsymbol{w}_i=(a_1,\ldots,\widehat{a_i},\ldots,a_4)$ for some $i=1,2,3,4$, where the symbol $\widehat{a_i}$ means that a_i is missing. Call such a term *odd* if i is odd and *even* if i is even. Put $\varepsilon(\boldsymbol{w}_i)=(-1)^i$. Next, given a vector $\boldsymbol{w}=(a_{i_1},a_{i_2},a_{i_3})$, where $1\le i_1<i_2<i_3\le 4$ let us associate to it the Fibonacci product $F_{\boldsymbol{w}}=F_{a_{i_1}-a_{i_2}}F_{a_{i_2}-a_{i_3}}F_{a_{i_3}-a_{i_1}}$.

Using that $F_{-n}=(-1)^{n-1}F_n$, it is then easy to see that the left hand side of (4) is just

$$(-1)^{a_2-a_4-1}F_{a_2-a_3}F_{a_3-a_4}F_{a_4-a_2}+(-1)^{a_2-a_4}F_{a_1-a_3}F_{a_3-a_4}F_{a_4-a_1}$$

$$+(-1)^{a_2-a_4-1}F_{a_1-a_2}F_{a_2-a_4}F_{a_4-a_1}+(-1)^{a_2-a_4}F_{a_1-a_2}F_{a_2-a_3}F_{a_3-a_1}=(-1)^{a_2-a_4}\sum_{i=1}^{4}\varepsilon(\boldsymbol{w}_i)F_{\boldsymbol{w}_i}. \tag{5}$$

For each $\boldsymbol{w}=\boldsymbol{w}_i$, we apply the lemma with x, y being the indices of the first two Fibonacci numbers appearing in $F_{\boldsymbol{w}}$ (note that the third index is always the negative of the sum of the first two) and get that the sum (5) is a sum of the type

$$\sum_{(i,j)\in\{1,2,3,4\}^2} c_{i,j}\alpha^{2a_i-2a_j},$$

where $c_{i,j}$ is some coefficient. It suffices to show that $c_{i,j}=0$ for all pairs (i,j). This is clear from the lemma for the pairs for which $i=j$. There are a total of 12 remaining pairs (i,j) with $i\neq j$. Using the lemma and the Binet's formula, we see that the sum appearing in the right hand side of (5) is a linear combination of $4\times 3\times 2=24$ expressions of the type $\alpha^{2a_i-2a_j}$ each with coefficient $\pm 1/5^{3/2}$. It is easy to check that each pair (i,j) appears twice with opposite signs, namely one in each of $\varepsilon(\boldsymbol{w}_t)F_{\boldsymbol{w}_t}$, where $t\in\{1,2,3,4\}\backslash\{i,j\}$, which does show that expression (5) is zero.

Example 1. Let $i=1,\ j=2$. If $t=3$, then

$$\varepsilon(\boldsymbol{w}_3)F_{\boldsymbol{w}_3}=-F_{a_1-a_2}F_{a_2-a_4}F_{a_4-a_1}=-\frac{(-1)^{a_1-a_2}}{5^{3/2}}\alpha^{2a_1-2a_2}+\dots,$$

while if $t=4$, then

$$\varepsilon(\boldsymbol{w}_4)F_{\boldsymbol{w}_4}=F_{a_1-a_2}F_{a_2-a_3}F_{a_3-a_1}=\frac{(-1)^{a_1-a_2}}{5^{3/2}}\alpha^{2a_1-2a_2}+\dots,$$

so we see that $\alpha^{2a_1-2a_2}$ appears twice with opposite signs, as promised.

We now look at (3). It is easy to see, by the same argument of changing signs consistently as in (5), that the left hand side of (3) is just

$$(-1)^{a_2-a_4}\sum_{i=1}^{4}(-1)^{a_i}\alpha^{2a_i}\varepsilon(\boldsymbol{w}_i)F_{\boldsymbol{w}_i}. \tag{6}$$

Using Binet's formula and the lemma, one checks again easily that the above sum (6) is zero. Indeed, by the lemma and Binet's formula again, the above sum is a formal linear combination of 24 expressions of the form $\alpha^{2a_i+2a_j-2a_k}$, where $\{i,j,k\}\subset\{1,2,3,4\}$ and $\{i,j,k\}$ has exactly 3 elements, with coefficients $\pm 1/5^{3/2}$. There are $\binom{4}{3}\cdot 3=12$ such expressions, so it again suffices to check that each such expression appears exactly twice with opposite signs; namely, $\alpha^{2a_i+2a_j-2a_k}$ appears from both $(-1)^{a_i}\alpha^{2a_i}\varepsilon(\boldsymbol{w}_i)F_{\boldsymbol{w}_i}$ as well as from $(-1)^{a_j}\alpha^{2a_j}\varepsilon(\boldsymbol{w}_j)F_{\boldsymbol{w}_j}$.

Example 2. Let $i = 1,\ j = 2,\ k = 3$. Then

$$(-1)^{a_1}\alpha^{2a_1}\varepsilon(\boldsymbol{w}_1)F_{\boldsymbol{w}_1} = -(-1)^{a_1}\alpha^{2a_1}F_{a_2-a_3}F_{a_3-a_4}F_{a_4-a_2}$$

$$= -\frac{(-1)^{a_1+(a_2-a_3)}}{5^{3/2}}\alpha^{2a_1+(2a_2-2a_3)} + \dots ,$$

while

$$(-1)^{a_2}\alpha^{2a_2}\varepsilon(\boldsymbol{w}_2)F_{\boldsymbol{w}_2} = (-1)^{a_2}\alpha^{2a_2}F_{a_1-a_3}F_{a_3-a_4}F_{a_4-a_1}$$

$$= \frac{(-1)^{a_2+(a_1-a_3)}}{5^{3/2}}\alpha^{2a_2+(2a_1-2a_3)} + \dots ,$$

so we see that $\alpha^{2a_1+2a_2-2a_3}$ appears twice with opposite signs, as promised.

Finally, it remains to prove (2). Using again relation (5), we get that the right hand side of formula (2) equals

$$(-1)^{a_2-a_4}\sum_{i=1}^{4}\alpha^{4a_i}\varepsilon(\boldsymbol{w}_i)F_{\boldsymbol{w}_i}. \tag{7}$$

By the lemma and Binet's formula, the above expression is of the form

$$\sum_{(i,j,k):\{i,j,k\}\subset\{1,2,3,4\}}\frac{c_{(i,j,k)}}{5^{3/2}}\alpha^{4a_i+2a_j-2a_k}.$$

Here, $c_{(i,j,k)} = 0$ if $\{i,j,k\}$ has less than 3 elements and $c_{(i,j,k)} = \pm 1$ otherwise. There are precisely 24 ordered triples (i,j,k) with distinct elements from $\{1,2,3,4\}$. To complete the proof, we proceed in two steps:

Assumption. *Assume that the right hand side of expression (2) is of the form*

$$\sum_{(i,j,k):\{i,j,k\}\subset\{1,2,3,4\}}\frac{c'_{(i,j,k)}}{5^{3/2}}\alpha^{4a_i+2a_j-2a_k},$$

where $c_{(i,j,k)} = 0$ *if* $\{i,j,k\}$ *has less than* 3 *elements and* $c'_{(i,j,k)} = \pm 1$ *otherwise.*

Proof under the Assumption: In this case, we simply show that $c_{(i,j,k)} = c'_{(i,j,k)}$ for all ordered triples of distinct elements (i,j,k). Up to relabeling the variables and changing some signs, we may assume that $i=1,\ j=2,\ k=3$. The coefficient of $\alpha^{4a_1+2a_2-2a_3}$ in the left hand side of (2) comes as part of the expression

$$\alpha^{4a_1}F_{a_2-a_3}F_{a_2-a_4}F_{a_3-a_4},$$

while in the right hand side it comes from

$$F_{a_2-a_3}F_{a_2-a_4}F_{a_3-a_4}\left(5\sqrt{5}F_{a_1-a_2}F_{a_1-a_3}F_{a_1-a_4}\alpha^{a_1+a_2+a_3+a_4}\right),$$

and it is clear by the Binet's formulas that

$$5\sqrt{5}F_{a_1-a_2}F_{a_1-a_3}F_{a_1-a_4}\alpha^{a_1+a_2+a_3+a_4} = \alpha^{4a_1} + \text{other powers of } \alpha,$$

where the other powers of α above are of the form $\alpha^{c_1a_1+c_2a_2+c_3+c_4a_4}$ with integers c_1, c_2, c_3, c_4 and $c_1 < 4$. This argument shows that $c_{(1,2,3)} = c'_{(1,2,3)}$, which completes the proof of this step.

It remains to prove the above Assumption. Let us take a look at the right hand side of expression (2). Let $\boldsymbol{E}$ be the set of all 6 dimensional vectors $\boldsymbol{e} = (e_{i,j})_{1\le i<j\le 3}$ with $e_{i,j} \in \{\pm 1\}$ for all $1 \le i < j \le 3$. For each such vector $\boldsymbol{e}$ let $\varepsilon(\boldsymbol{e}) = \prod_{1\le i<j\le 3} e_{i,j}$. Define also $e_{j,i} = -e_{i,j}$ for $j > i$ both in $\{1,2,3,4\}$, and $e_{i,i} = 0$ for all $i \in \{1,2,3,4\}$, so that $e_{i,j}$ is defined for all $i,j \in \{1,2,3,4\}$. Given a four dimensional vector $\boldsymbol{a} = (a_1,a_2,a_3,a_4)$, let

$$\Delta(\boldsymbol{a}) = (a_1-a_2, a_1-a_3, a_1-a_4, a_2-a_3, a_2-a_4, a_3-a_4),$$

and put

$$t(\boldsymbol{e},\boldsymbol{a}) = a_1+a_2+a_3+a_4+\langle \boldsymbol{e}, \Delta(\boldsymbol{a})\rangle = a_1t_1(\boldsymbol{e}) + a_2t_2(\boldsymbol{e}) + a_3t_3(\boldsymbol{e}) + a_4t_4(\boldsymbol{e}),$$

where

$$t_i(\boldsymbol{e}) = 1 + \sum_{j=1}^{4} e_{j,i} \qquad \text{for all } i = 1,2,3,4.$$

Using Binet's formula under the form

$$F_x = \frac{1}{\sqrt{5}}\Big(\sum_{\varepsilon\in\{\pm 1\}} (\varepsilon\alpha)^{\varepsilon x}\Big),$$

one checks easily that the right hand side of (2) is

$$\frac{1}{5\sqrt{5}} \sum_{\boldsymbol{e}\in \boldsymbol{E}} \varepsilon(\boldsymbol{e})\alpha^{\langle t(\boldsymbol{e},\boldsymbol{a})\rangle}. \tag{8}$$

Write $d_i = t_i(\boldsymbol{e})$. Thus, $\langle t(\boldsymbol{e}), \boldsymbol{a}\rangle = d_1a_1 + d_2a_2 + d_3a_3 + d_4a_4$. To complete the proof of the Assumption, we shall show that if $\boldsymbol{d} = (d_1, d_2, d_3, d_4)$ is a permutation on $(4, 2, -2, 0)$, then there exists a unique $\boldsymbol{e}$ such that $t_i(\boldsymbol{e}) = d_i$ for $i = 1, 2, 3, 4$. We shall also show that if $\boldsymbol{d}$ is not a permutation of $(4, 2, -2, 0)$, then there exists an even number of $\boldsymbol{e}$ such that $t_i(\boldsymbol{e}) = d_i$ for $i = 1, 2, 3, 4$, and that if this even number is positive, then half of those $\boldsymbol{e}$ will have $t(\boldsymbol{e}) = 1$ and the other half will have $t(\boldsymbol{e}) = -1$. It is clear that if we prove the above claims, then the Assumption will follow from them and formula (8) for the right hand side of (2).

To prove the above claims, note that $-3 \le d_i \le 4$ and d_i is even for all $i = 1, 2, 3, 4$. Note further that $\sum_{i=1}^{4} d_i = 4$. Further, if $d_i = 4$ for some i, then $d_j < 4$ for all $j \ne i$.

Assume say that $d_1 = 4$. Then $e_{1,2} = e_{1,3} = e_{1,4} = 1$. Further, $d_i \in \{-2, 0, 2\}$ for $i = 2, 3, 4$, and their sum is zero.

Assume that there exists $i > 1$ such that $d_i \ne 0$. Then there must exist $i > 1$ such that $d_i > 0$. Assume that $d_2 = 2$. Then $\{d_3, d_4\} = \{0, -2\}$. Assume, say that $d_3 = 0$, $d_4 = -2$. Hence, $\boldsymbol{d} = (4, 2, 0, -2)$. Then one checks easily that the only $\boldsymbol{e}$ is given by $e_{i,j} = 1$ for all $1 \le i < j \le 3$.

Assume now still that $d_1 = 4$, but $d_2 = d_3 = d_4 = 0$. We then get that $e_{1,2} = e_{1,3} = e_{2,3} = 1$ and $e_{2,3} = -e_{2,4} = e_{3,4}$. Putting $e_{2,3} = \varepsilon$ for some $\varepsilon \in \{\pm 1\}$, we get that $t(\boldsymbol{e}) = -\varepsilon$. Hence, if $\boldsymbol{d} = (2, 0, 0, 0)$, then there exist two values of $\boldsymbol{e}$ such that $t_i(\boldsymbol{e}) = d_i$ for $i = 1, 2, 3, 4$, and their corresponding $\varepsilon(\boldsymbol{e})$ have opposite signs.

Assume now that $d_i < 4$ for all $i = 1, 2, 3, 4$. Since the sum of the d_i's is 4, we get that at least two of them are 2. So, either three of them are 2 and the remaining one is -2, or two of them are 2 and the other two are 0.

Assume that $d_1 = d_2 = d_3 = 2$ and $d_4 = -2$. Looking at $d_4 = -2$, we get that $e_{1,4}$ cannot be -1, so $e_{1,4} = 1$. The same equation now implies that $e_{2,4} = e_{3,4} = 1$, and now the relations $d_i = 2$ for $i = 1, 2, 3$ give $e_{1,2} = -e_{1,3} = e_{2,3}$. Thus, if we put $e_{1,2} = \varepsilon \in \{\pm 1\}$, then $\varepsilon(\boldsymbol{e}) = \varepsilon$. Hence, we showed that if $\boldsymbol{d} = (2, 2, 2, -2)$, then there exist precisely two $\boldsymbol{e}$ with the property that $t_i(\boldsymbol{e}) = d_i$ for $i = 1, 2, 3, 4$, and further their $\varepsilon(\boldsymbol{e})$ have opposite signs.

Finally, assume that $\boldsymbol{d} = (2, 2, 0, 0)$. In this case, one checks that there are precisely four values of $\boldsymbol{e}$, namely $\boldsymbol{e} = (1, \varepsilon, -\varepsilon, 1, 1, \varepsilon)$, $(-1, 1, 1, \varepsilon, -\varepsilon, \varepsilon)$ for $\varepsilon \in \{\pm 1\}$, and it is clear that of the four numbers $t(\boldsymbol{e})$ for the above $\boldsymbol{e}$, two of them are $+1$ and the other two are -1.

This completes the proof the claims; hence, of the Assumption and consequently completes the solution to the problem.

Note: No other solution was received. In a correspondence dated September 1, 2005, Paul S. Bruckman mentions that he and Ray Melham had also solved and generalized H614. These general results will appear in a forthcoming paper in the *Fibonacci Quarterly.* The solution of K. Davenport on which this solution is based is dated October 17, 2005.