ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Russ Euler and Jawad Sadek

Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by July 15, 2007. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2,\ \beta = (1-\sqrt{5})/2,\ F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1021 Proposed by Harris Kwong, SUNY, Fredonia, Fredonia, NY

Prove that, for any integers m and N,

$$N^4 - 2L_mN^3 + 4F_{m+1}F_{m-1}N^2 + 2F_m^2L_mN - F_{2m}^2$$

is of the form $X^2(Y^2 - Z^2)$ for some integers $X,\ Y$ and Z.

B-1022 Proposed by Paul S. Bruckman, Sointula, Canada

If $u_n = F_{n+1}F_n$, and $x_n = \{3u_n + (-1)^n\}\{u_n - (-1)^n\}$, prove the following idenitity, for all integers n:

$$(x_n)^2 + (4u_{n+1}u_{n-1})^2 = (F_{2n+1})^4.$$

B-1023 Proposed by Paul S. Bruckman, Sointula, Canada

Prove the following identity, for all integers n:

$$(F_n u_{n+2})^2 + (F_{n+1} u_{n-1})^2 = (F_{2n+1})^3,$$

where $u_n = (F_n)^2 + 2(-1)^n$.

B-1024 Proposed by José Luis Díaz-Barrero and Miquel Grau-Sánchez, Universidad Politécnica de Cataluña, Barcelona, Spain

Let n be a positive integer. Prove that

$$F_{n-1} \leq \frac{1}{2}\left(\sqrt[n]{\prod_{k=1}^{n}(k+L_n)} - \sqrt[n]{\prod_{k=1}^{n}(k+F_n)}\right).$$

B-1025 Proposed by José Luis Díaz-Barrero, Barcelona Spain and Pantelimon George Popescu, Bucharest, Romania

Let n be a positive integer. Prove that

$$\prod_{k=1}^{n} F_k^{(F_k+F_{2n})C(n,k)} \leq \left(\prod_{k=1}^{n} F_k^{F_k C(n,k)}\right)^{2^n}$$

where $C(n,k)$, for all $k \in \{1, 2, \ldots, n\}$, is the binomial coefficient $\binom{n}{k}$.

SOLUTIONS

An Identity Problem!

B-1010 Proposed by Stanley Rabinowitz, MathPro Press, Chelmsford, MA (Vol. 44, no. 1, February 2006)

Find positive integers a, b, and m (with $m > 1$) such that

$$F_n \equiv b^n - a^n \pmod{m}$$

is an identity (i.e., true for all n) or prove that no identity of this form exists.

Solution by John Morrison, Baltimore, MD

There is no identity of the form (*) $F_n \equiv b^n - a^n \pmod{m}$ with positive integers a, b, $m (m > 1)$.

First: Note that if a congruence holds for some m then it holds for any prime p dividing m, so it is enough to show (*) can not hold for any prime p.

Next: If $p = 2$, since $0^n = 0$ and $1^n = 1$, for any fixed a and b, $b^n - a^n$ is constant for all n and so not equal to the sequence F_n mod 2, which is $(1, 1, 0, \dots)$ for $n \geq 1$.

Finally: If $p > 2$, letting $n = 1$ we see that $b - 1 \equiv 1 \pmod{p}$ and letting $n = 2$ we see that $b^2 - a^2 \equiv (b - a)(b + a) \equiv (b + a) \equiv 1 \pmod{p}$. Adding $b - a \equiv 1 \pmod{p}$ and $b + a \equiv 1 \pmod{p}$ gives $2b \equiv 2 \pmod{p}$ and so, since 2 has an inverse mod p, we have $b \equiv 1 \pmod{p}$. Then $a \equiv 0 \pmod{p}$ and $b^n - a^n \equiv b^n \equiv 1 \pmod{p}$ for all n. This is not equal to F_n mod p, which is $(1, 1, 2, \dots)$ for $n \geq 1$ and $p > 2$.

Also solved by Paul S. Bruckman, Russell J. Hendel, Harris Kwong, David Maner, H.-J. Seiffert, James A. Sellers, David Stone and John Hawking (jointly), and the proposer.

A Binomial Type Identity

B-1011 Correction **Proposed by José Luis Díaz-Barrero, Universidad Politécnica de Cataluña, Barcelona, Spain (Vol. 44, no. 2, May 2006)**

Let n be a positive integer. Prove that

$$(F_n)^5 + (L_n)^5 = 2F_{n+1}\{16(F_{n+1})^4 - 20(F_{n+1})^2 F_{2n} + 5(F_{2n})^2\}.$$

Solution by E. Gokcen Alptekin, Seluck University, Konya, Turkey

It is known that $F_n + L_n = 2F_{n+1}$. Taking $L_n = 2F_{n+1} - F_n$,

$$\begin{aligned}
F_n^5 + L_n^5 &= F_n^5 + (2F_{n+1} - F_n)^5 \\
&= 32F_{n+1}^5 - 80F_{n+1}^4F_n + 80F_{n+1}^3F_n^2 - 40F_{n+1}^2F_n^3 + 10F_{n+1}F_n^4 \\
&= 32F_{n+1}^5 + 10F_nF_{n+1}(F_n^3 - 4F_n^2F_{n+1} + 8F_nF_{n+1}^2 - 8F_{n+1}^3) \\
&= 32F_{n+1}^5 + 10F_nF_{n+1}(F_n^3 - 6F_n^2F_{n+1} + 12F_nF_{n+1}^2 - 8F_{n+1}^3 + 2F_n^2F_{n+1} - 4F_nF_{n+1}^2) \\
&= 32F_{n+1}^5 + 10F_nF_{n+1}\left((F_n - 2F_{n+1})^3 + 2F_nF_{n+1}(F_n - 2F_{n+1})\right) \\
&= 32F_{n+1}^5 + 10F_nF_{n+1}\left((F_n - 2F_{n+1})(F_n^2 - 4F_nF_{n+1} + 4F_{n+1}^2 + 2F_nF_{n+1})\right) \\
&= 32F_{n+1}^5 + 10F_nF_{n+1}\left(-L_n(F_n^2 - 2F_nF_{n+1} + 4F_{n+1}^2)\right) \\
&= 32F_{n+1}^5 - 40F_nL_nF_{n+1}^3 + 10F_nF_{n+1}(-L_nF_n(F_n - 2F_{n+1})) \\
&= 32F_{n+1}^5 - 40F_nL_nF_{n+1}^3 + 10F_nF_{n+1}(L_nF_nL_n).
\end{aligned}$$

Using $F_nL_n = F_{2n}$, we obtain

$$\begin{aligned}
F_n^5 + L_n^5 &= 32F_{n+1}^5 - 40F_{2n}F_{n+1}^3 + 10F_{n+1}(F_nL_n)^2 \\
&= 2F_{n+1}(16F_{n+1}^4 - 20F_{n+1}^2F_{2n} + 5F_{2n}^2).
\end{aligned}$$

Also solved by Paul S. Bruckman, Kenneth B. Davenport, Ovidiu Furdui, G. C. Greubel, Ralph P. Grimaldi, Harris Kwang, H.-J. Seiffert, and the proposer.

A Sum of ArcTangents

B-1012 Proposed by Br. J. Mahon, Australia (Vol. 44, no. 1, February 2006)

Prove that

$$\sum_{i=1}^{n} \tan^{-1}\frac{1}{F_{2i-1}} = \tan^{-1} F_{2n}.$$

Solution by Charles K. Cook, University of South Carolina Sumter, Sumter, SC

This problem is a slight variation of work appearing in paragraph 8 of [1].

Let $\tan\theta_n = \frac{1}{F_n}$. Then

$$\tan(\theta_{2n-2} - \theta_{2n}) = \frac{\tan\theta_{2n-2} - \tan\theta_{2n}}{1 + \tan\theta_{2n-2}\tan\theta_{2n}} = \frac{\frac{1}{F_{2n-2}} - \frac{1}{F_{2n}}}{1 + \frac{1}{F_{2n-2}F_{2n}}}$$

$$= \frac{F_{2n} - F_{2n-2}}{F_{2n-2}F_{2n} + 1} = \frac{F_{2n-1}}{F_{2n-1}^2} = \frac{1}{F_{2n-1}}. \left(\text{Since } F_{2n-2}F_{2n} - F_{2n-1}^2 = (-1)^{2n-1}.\right)$$

Thus

$$\sum_{i=1}^{n} \tan^{-1} \frac{1}{F_{2i-1}} = \sum_{i=1}^{n} \left(\tan^{-1} \frac{1}{F_{2i-2}} - \tan^{-1} \frac{1}{F_{2i}} \right) = \frac{\pi}{2} - \tan^{-1} \frac{1}{F_{2n}} = \tan^{-1} F_{2n}.$$

References.

1. "A Primer for the Fibonacci Sequence - Part V." *The Fibonacci Quarterly*, **2.1** (1964): 62-65.

Also solved by Gokcen Alptekin, Paul S. Bruckman, José Luis Díaz-Barrero and Miquel Grau-Sanchez (jointly), Ovidiu Furdui, G.C. Greubel, Ralph P. Grimaldi, Russell J. Hendel, Harris Kwong, David Maner, John F. Morrison, H.-J. Seiffert, James Sellers, David Stone and John Hawkins (jointly), and the proposer.

A Convex Inequality

B-1013 Proposed by the Solution Editor (Vol. 44, no. 1, February 2006)

Prove that

$$\frac{F_n^{2^n} F_{n+1}^{2^n}}{n^{2^n-1}} \leq F_1^{2^{n+1}} + F_2^{2^{n+1}} + \cdots + F_n^{2^{n+1}}$$

for all integers $n \geq 1$.

Solution by Ovidiu Furdui, Western Michigan University, Kalamazoo, MI

The well known formula

$$F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}$$

will be used.

In view of the convexity of the function $x \to x^{2^n}$ on $(0, \infty)$ we get that

$$\left(\frac{F_1^2 + F_2^2 + \cdots + F_n^2}{n} \right)^{2^n} \leq \frac{F_1^{2^{n+1}} + F_2^{2^{n+1}} + \cdots + F_n^{2^{n+1}}}{n}.$$

Equivalently

$$\frac{(F_n F_{n+1})^{2^n}}{n^{2^n-1}} \leq F_1^{2^{n+1}} + F_2^{2^{n+1}} + \cdots + F_n^{2^{n+1}}.$$

We are done.

Also solved by Paul S. Bruckman, José Luis Díaz-Barrero and José Gibergans-Báguena (jointly), Russell J. Hendel, Harris Kwong, H.-J. Seiffert, and the proposer.

A Geometric Series

B-1014 (Correction) **Proposed by Steve Edwards, Southern Polytechnic State University, Marietta, GA (Vol. 44, no. 2, May 2006)**

Show that $\sum_{k=0}^{\infty} \alpha^{-nk}$ equals $\frac{F_n\alpha+F_{n-1}-1}{L_n-2}$ if n is a positive even integer, and $\frac{F_n\alpha+F_{n-1}+1}{L_n}$ if n is a positive odd integer.

Solution by Roger Haskell, Scotia, New York

Since $\alpha^{-n} < 1$, the given series converges to $\frac{\alpha^n}{\alpha^n-1}$.

Noting $\alpha \cdot \beta = -1$, and multiplying numerator and denominator by $(\alpha^n + 1)$ with n odd, gives

$$\frac{(\alpha^n + 1) \cdot \alpha^n}{\alpha^{2 \cdot n} + (\alpha \cdot \beta)^n} = \frac{\alpha^n + 1}{L_n}.$$

Multiplying numerator and denominator by $(\alpha^n - 1)$ with n even, gives

$$\frac{(\alpha^n - 1) \cdot \alpha^n}{\alpha^{2 \cdot n} - 2 \cdot \alpha^n + (\alpha \cdot \beta)^n} = \frac{\alpha^n - 1}{L_n - 2}.$$

With $\alpha^n = \alpha \cdot F_n + F_{n-1}$, the desired result is obtained. That this is the case, follows from Binet's formula and the definition of α. Thus

$$\frac{\alpha^{n+1} + \alpha^{n-1} - (\alpha \cdot \beta) \cdot \beta^{n-1} - \beta^{n-1}}{\sqrt{5}} = \alpha^n \cdot \left(\frac{\alpha^2 + 1}{\alpha \cdot \sqrt{5}} \right) = \alpha^n.$$