# ADVANCED PROBLEMS AND SOLUTIONS

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## PROBLEMS PROPOSED IN THIS ISSUE

**H-725 Proposed by Paul S. Bruckman, Nanaimo, BC**

Prove the following identities valid for $n = 0, 1, 2, \ldots$

(a) $$\sum_{k=0}^{\lfloor n/4\rfloor}\binom{n-3k}{k}(-3)^{3k}4^{n-4k} = \frac{1}{6}\left((3n+5)3^n - (-1)^n 3^{n/2}\frac{\sin((n-1)\theta)}{\sin\theta}\right),$$
where $\sin\theta = \sqrt{2/3}$;

(b) $$\sum_{k=0}^{\lfloor n/4\rfloor}\binom{n-k}{3k}(-3)^{3k}4^{n-4k} = \frac{1}{18}\left(9n+7+3^{3n/2}(11\cos(n\rho)+\sin(n\rho)/\sqrt{2})\right),$$
where $\sin\rho = \sqrt{2/27}$;

(c) $$\sum_{k=0}^{\lfloor n/4\rfloor}\binom{n-2k}{2k}\left(\frac{(pq(p^2-q^2))}{(p^2+q^2)^2}\right)^{2k} = \frac{(p(p+q))^{n+1}-(q(q-p))^{n+1}}{2(p^2+2pq-q^2)(p^2+q^2)^n}$$
$$+\frac{(p(p-q))^{n+1}-(q(q+p))^{n+1}}{2(p^2-2pq-q^2)(p^2+q^2)^n},$$ where $p > q > 0$ are integers.

**H-726 Proposed by Hideyuki Ohtsuka, Saitama, Japan**

Prove that
$$\sum_{k=1}^{\infty}\left(\frac{1}{F_{2k}}-\frac{1}{F_{4k}}+\frac{1}{F_{8k}}+\frac{1}{F_{16k}}+\cdots+\frac{1}{F_{2^n k}}+\cdots\right) = \sum_{k=1}^{\infty}\frac{1}{F_{2k-1}F_{2k}}.$$

**H-727 Proposed by Bassem Ghalayini, Louaize, Lebanon**

Let $n$ be a natural number. Prove that

$$(2n+1)\binom{2n}{n} = \sum_{\substack{0\le i,j,k\le n \\ i+j+k=n}} \binom{2i}{i}\binom{2j}{j}\binom{2k}{k}.$$

**H-728 Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania**

Let $a, b, c, m$ be positive real numbers and $n$ be a positive integer. Prove that:

(a) $\dfrac{F_n}{\sqrt{F_n^2 + aF_{n+1}F_{n+2}}} + \dfrac{F_{n+1}}{\sqrt{F_{n+1}^2 + bF_{n+2}F_n}} + \dfrac{F_{n+2}}{\sqrt{F_{n+2}^2 + cF_nF_{n+1}}} \ge 1,$
provided that $a + b + c \le 24$;

(b) $\dfrac{a^{-3m-3}}{(F_nb + F_{n+1}c)^{m+1}} + \dfrac{b^{-3m-3}}{(F_nc + F_{n+1}a)^{m+1}} + \dfrac{c^{-3m-3}}{(F_na + F_{n+1}b)^{m+1}} \ge \dfrac{3}{F_{n+2}^{m+1}},$
provided that $abc = 1$.

**H-729 Proposed by Paul S. Bruckman, Nanaimo, BC**

Define a sequence $\{a_n\}_{n\ge 0}$ of rational numbers by the recurrence $\displaystyle\sum_{k=0}^{n} \frac{a_k}{n+1-k} = \delta_{n,0}$, where $\delta_{i,j}$ is the Kronecker symbol which equals 1 if $i = j$ and 0, otherwise.

(a) Prove that $-\displaystyle\sum_{k=1}^{\infty} \frac{a_n}{n} = \gamma$, the Euler constant;

(b) Prove that $a_n = -\dfrac{1}{n+1} + \displaystyle\sum_{k=0}^{n-1} u_{n-k}a_k$ for $n \ge 1$, where $u_m = \dfrac{2(H_m - 1)}{(m+2)}$ and $H_m = \displaystyle\sum_{k=1}^{m} \frac{1}{k}$ for all $m \ge 1$.

## SOLUTIONS

### On the Parity of the Mertens Function

**H-700 Proposed by Mohamed El Bachraoui, United Arab Emirates (Vol. 48, No. 2, May 2011)**

Let $\mu$ be the Möbius mu function and let $M(n)$ be the Mertens function given by $M(n) = \sum_{a\le n} \mu(a)$. If $n > 2$, it is clear that

$$M(n) \equiv \#\{a \in [2, n-1] : \ a \text{ squarefree}, \ a \nmid n\} \pmod 2.$$

Prove that for all positive integers $n > 2$ we have

a) $M(2n) \equiv 1 + \#\{a \in [2, 2n-3] : \ a \text{ squarefree}, \ a \nmid 2n, \ a \nmid 2n-1, \ a \nmid 2n-2\} \pmod 2$;
b) $M(2n+1) \equiv \#\{a \in [2, 2n-2] : \ a \text{ squarefree}, \ a \nmid 2n+1, \ a \nmid 2n, a \nmid 2n-1\} \pmod 2$.

**Solution by the proposer.**

Let $n > 2$. By Theorem 10 in [1], we have

$$M(2n) = -3 + \sum_{a=1}^{2n} \mu(a) 2^{\lfloor 2n/a \rfloor - \lfloor (2n-3)/a \rfloor};$$

$$M(2n+1) = -4 + \sum_{a=1}^{2n+1} \mu(a) 2^{\lfloor (2n+1)/a \rfloor - \lfloor (2n-2)/a \rfloor}.$$

Then

$$M(2n) \equiv -3 + \sum_{\substack{1 \le a \le 2n \\ a \text{ is squarefree}}} \mu(a) 2^{\lfloor 2n/a \rfloor - \lfloor (2n-3)/a \rfloor} \equiv 1 + \sum_{\substack{1 \le a \le 2n \\ a \text{ is squarefree} \\ \lfloor 2n/a \rfloor = \lfloor (2n-3)/a \rfloor}} 1$$

$$\equiv 1 + \#\{a \in [2, 2n-3]; a \text{ square-free}, \ a \nmid 2n, \ a \nmid (2n-1), \text{ and } a \nmid (2n-2)\} \pmod 2.$$

The other identity follows similarly.

## References

[1] M. El Bachraoui, *Combinatorial identities involving Mertens function through relatively prime subsets*, arXiv: 0912.1518.

**Also solved by Paul S. Bruckman**.

## A Catalan Type Identity for $k$ Fibonacci Numbers

## H-701 Proposed by Sergio Falcón and Ángel Plaza, Gran Canaria, Spain (Vol. 48, No. 2, May 2011)

For $k \geq 1$, let $F_{k,n}$ be the sequence given by $F_{k,0} = 0, \ F_{k,1} = 1, \ F_{k,n+2} = kF_{k,n+1} + F_{k,n}$ for $n \geq 0$. Show that if $2r + h \neq 0$, then

$$\frac{F_{k,n+r}F_{k,n+r+h} + (-1)^{h+1}F_{k,n-r}F_{k,n-r-h}}{F_{k,2r+h}} = F_{k,2n}.$$

**Solution by Jallisa Clifford, Kristopher Liggins and Dickson Toroitich, Benedict College, SC.**

For simplicity of notation, we use the notation $F_{k,n} = F_n$ and hence, we will prove the following

$$F_{n+r}F_{n+r+h} + (-1)^{h+1}F_{n-r}F_{n-r-h} = F_{2r+h}F_{2n}.$$

If $h$ is even, let $h = 2a$. Recall Catalan's identity for the $k$-Fibonacci numbers [1] is $F_{n-r}F_{n+r} - F_n^2 = (-1)^{n+1-r}F_r^2$. By replacing $n = n + r + a$ and $r = a$, we obtain $F_{n+r}F_{n+r+2a} = F_{n+r+a}^2 + (-1)^{n+r+1}F_a^2$ and by replacing $n = n - r - a$ and $r = a$, we obtain $F_{n-r}F_{n-r-2a} = F_{n-r-a}^2 + (-1)^{n-r-2a+1}F_a^2$. Then

$$\begin{aligned}
&F_{n+r}F_{n+r+2a} + (-1)^{2a+1}F_{n-r}F_{n-r-2a} \\
&= [F_{n+r+a}^2 + (-1)^{n+r+1}F_a^2] - [F_{n-r-a}^2 + (-1)^{n-r-2a+1}F_a^2] \\
&= (F_{n+r+a}^2 - F_{n-r-a}^2) + (-1)^{n-r-2a+1}F_a^2((-1)^{2r+2a} - 1) \\
&= F_{n+r+a}^2 - F_{n-r-a}^2 = F_{2r+2a}F_{2n}.
\end{aligned}$$

This takes care of the case $h$ even. If $h$ is odd, let $h = 2a+1$. We prove first the following lemma.

**Lemma 1.**

$$F_{n+r}F_{n+r+1} + F_{n-r}F_{n-r-1} = F_{2r+1}F_{2n}.$$

*Proof of the Lemma:*

$$\begin{aligned}
LHS &= \frac{r_1^{n+r}-r_2^{n+r}}{r_1-r_2}\cdot\frac{r_1^{n+r+1}-r_2^{n+r+1}}{r_1-r_2} + \frac{r_1^{n-r}-r_2^{n-r}}{r_1-r_2}\cdot\frac{r_1^{n-r-1}-r_2^{n-r-1}}{r_1-r_2}\\
&= \frac{r_1^{2n+2r+1}-r_1^{n+r}r_2^{n+r+1}-r_1^{n+r+1}r_2^{n+r}+r_2^{2n+2r+1}}{(r_1-r_2)^2}\\
&\quad + \frac{r_1^{2n-2r-1}-r_1^{n-r}r_2^{n-r-1}-r_1^{n-r-1}r_2^{n-r}+r_2^{2n-2r-1}}{(r_1-r_2)^2}\\
&= \frac{r_1^{2n+2r+1}+r_2^{2n+2r+1}+r_1^{2n-2r-1}+r_2^{2n-2r-1}-(r_1+r_2)(r_1^{n+r}r_2^{n+r}+r_1^{n-r-1}r_2^{n-r-1})}{(r_1-r_2)^2}\\
&= \frac{r_1^{2n+2r+1}+r_2^{2n+2r+1}+r_1^{2n-2r-1}+r_2^{2n-2r-1}-(r_1+r_2)(r_1r_2)^{n-r-1}((r_1r_2)^{2r+1}+1)}{(r_1-r_2)^2}\\
&= \frac{r_1^{2n+2r+1}+r_2^{2n+2r+1}+r_1^{2n-2r-1}+r_2^{2n-2r-1}}{(r_1-r_2)^2}\\
&= \frac{r_1^{2n+2r+1}+r_2^{2n+2r+1}-r_1^{2n-2r-1}(r_1r_2)^{2r+1}-r_2^{2n-2r-1}(r_1r_2)^{2r+1}}{(r_1-r_2)^2}\\
&= \frac{r_1^{2n+2r+1}+r_2^{2n+2r+1}-r_1^{2n}r_2^{2r+1}-r_1^{2r+1}r_2^{2n}}{(r_1-r_2)^2} = \frac{(r_1^{2n}-r_2^{2n})(r_1^{2r+1}-r_2^{2r+1})}{(r_1-r_2)(r_1-r_2)}\\
&= F_{2n}F_{2r+1} = RHS,
\end{aligned}$$

since $r_1r_2 = -1$.

Next we will proceed by induction on $h$. If $h = 1$, the result follows from the previous lemma. We assume the identity is true for $h = 2a-1$; i.e.,

$$F_{n+r}F_{n+r+(2a-1)} + F_{n-r}F_{n-r-(2a-1)} = F_{2n}F_{2r+2a-1}.$$

Also note that the identity is true whenever $h$ is even; i.e.,

$$F_{n+r}F_{n+r+2a} - F_{n-r}F_{n-r-2a} = F_{2n}F_{2r+2a}.$$

We also use the recursive definition for the $k$-Fibonacci numbers,

$$F_{n+r+2a+1} = kF_{n+r+2a} + F_{n+r+2a-1} \quad \text{and} \quad F_{n-r-2a+1} = kF_{n-r-2a} + F_{n-r-2a-1}.$$

Thus,

$$\begin{aligned}
&F_{n+r}F_{n+r+2a+1} + (-1)^{2a+1+1}F_{n-r}F_{n-r-2a-1}\\
&= F_{n+r}(kF_{n+r+2a}+F_{n+r+2a-1}) + F_{n-r}(F_{n-r-2a+1}-kF_{n-r-2a})\\
&= k(F_{n+r}F_{n+r+2a}-F_{n-r}F_{n-r-2a}) + (F_{n+r}F_{n+r+2a-1}+F_{n-r}F_{n-r-2a+1})\\
&= kF_{2n}F_{2r+2a} + F_{2n}F_{2r+2a-1} = F_{2n}(kF_{2r+2a}+F_{2r+2a-1}) = F_{2n}F_{2r+2a+1}.
\end{aligned}$$

## References

[1] S. Falcón & Á. Plaza, *The k-Fibonacci sequence and the Pascal 2-triangle*, Chaos, Solitons and Fractals, **33** (2007), 38–49.


**Also solved by Paul S. Bruckman, Zbigniew Jakubczyk and the proposers.**

Sums of Reciprocals of Squares of Lucas Numbers

**H-702 Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 48, No. 2, May 2011)**

For an integer $m \neq 0$ determine

$$\sum_{k=1}^{\infty} \frac{4^k}{L_{m2^k}^2}.$$

**Solution by Ángel Plaza, Las Palmas, Spain and Derek Jennings, Southampton, England.**

The proof for the case $m = 1$ is in [1]. We follow the same argument given there for the general case. We start from the identities

$$\frac{q^m}{(1+q^m)^2} + \frac{4q^{2m}}{(1-q^{2m})^2} = \frac{q^m}{(1-q^m)^2}$$

$$\frac{q^m}{(1+q^m)^2} + \frac{4q^{2m}}{(1-q^{2m})^2} + \frac{16q^{4m}}{(1-q^{4m})^2} = \frac{q^m}{(1-q^m)^2}$$

and continuing the expansion process we arrive at

$$\frac{q^m}{(1+q^m)^2} + \sum_{n=1}^{\infty} \frac{2^{2n} q^{m2^n}}{(1+q^{m2^n})^2} = \frac{q^m}{(1-q^m)^2}.$$

$$\sum_{n=1}^{\infty} \frac{2^{2n} q^{m2^n}}{(1+q^{m2^n})^2} = 4 \cdot \frac{q^{2m}}{(1-q^{2m})^2}. \tag{1}$$

Now setting $q = (1-\sqrt{5})/2$, and using the Binet formulas $L_n = \alpha^n + \beta^n$, and $F_n = (\alpha^n - \beta^n)/(\alpha - \beta)$, we obtain for $n = 1$ and $n > 1$, respectively

$$\left(\frac{2q^m}{1+q^{2m}}\right)^2 = \left(\frac{2}{q^{-m}+q^m}\right)^2 = \frac{4}{(q^{-m}+q^m)^2},$$

$$\frac{q^{m2^n}}{(1+q^{m2^n})^2} = \left(\frac{q^{m2^{n-1}}}{1+q^{m2^n}}\right)^2 = \left(\frac{1}{q^{-m2^{n-1}}+q^{m2^{n-1}}}\right)^2 = \frac{1}{L_{m2^{n-1}}^2}.$$

Equation (1) reads as

$$\frac{4}{(q^{-m}+q^m)^2} + 4\sum_{n=1}^{\infty} \left(\frac{2^n}{L_{m2^n}}\right)^2 = 4 \cdot \frac{q^{2m}}{(1-q^{2m})^2}, \tag{2}$$

from where we derive

$$\sum_{n=1}^{\infty} \left(\frac{2^n}{L_{m2^n}}\right)^2 = \frac{q^{2m}}{(1-q^{2m})^2} - \frac{1}{(q^{-m}+q^m)^2} = \frac{4}{5F_{2m}^2}.$$

## References

[1] D. Jennings, *Some reciprocal summation identities with applications to the Fibonacci and Lucas numbers*, Applications of Fibonacci Numbers, Vol. 7, Edition 1, G. E. Bergum, Alwyn F. Horadam, A. N. Philippou (eds.), Kluwer Acad. Publ., 1998.

**Also solved by Paul S. Bruckman and the proposer.**

## Binomial Coefficients and Fibonacci and Lucas Numbers

**H-703 Proposed by Napoleon Gauthier, Kingston, ON (Vol. 48, No. 2, May 2011)**

Let $n$ be a positive integer and prove the following identities:

a) $\displaystyle\sum_{k\geq 0} k\binom{n-k-1}{k} = \frac{1}{10}[(5n-4)F_n - L_n]$;

b) $\displaystyle\sum_{k\geq 0} k^2\binom{n-k-1}{k} = \frac{1}{50}[(15n^2-20n+4)F_n - (5n^2-6n)L_n]$.

**Solution by Ángel Plaza and Sergio Falcón, Las Palmas, Spain.**

a) For the first values of $n$ in both hands of the equality we obtain the sequence

$$\{0, 0, 1, 2, 5, 10, 20, 38, 71, 130, 235, 420, \ldots\}$$

listed in [1] as sequence $A001629$, and called as *Fibonacci numbers convolved with themselves.* In order to prove the equality, we show that both sides of the equality have the same generating function.

For the right-hand side, we use the "Snake Oil Method" [2] applied to

$$a_n = \sum_{k\geq 0} k\binom{n-k-1}{k}.$$

Let $A(x)$ be its generating function. That is,

$$A(x) = \sum_{n\geq 0} x^n \sum_{k\geq 0} k\binom{n-k-1}{k} = \sum_{k\geq 0} kx^{k+1} \sum_{n\geq 2k+1} \binom{n-k-1}{k} x^{n-k-1}.$$

We use the following identity (see Eq. (4.3.1), page 120 in [2]):

$$\sum_{r\geq 0} \binom{r}{k} x^r = \frac{x^k}{(1-x)^{k+1}} \qquad (k \geq 0).$$

Therefore,

$$A(x) = \sum_{k\geq 0} kx^{k+1} \frac{x^k}{(1-x)^{k+1}} = \frac{x}{1-x} \sum_{k\geq 0} k\left(\frac{x^2}{1-x}\right)^k = \frac{x^3}{(-1+x+x^2)^2}.$$

In the last step, we used $\displaystyle\sum_{k\geq 0} ky^k = \frac{y}{(1-y)^2}$.

Now, we obtain the generating function of the sequence on the right-hand side of the equality. Since $F(x) = \dfrac{x}{1-x-x^2}$ and $L(x) = \dfrac{2-x}{1-x-x^2}$ are the generating functions for Fibonacci and Lucas numbers, respectively, the generating function for $\{nF_n\}_{n\geq 0}$ is

$$xF'(x) = \frac{x+x^3}{(-1+x+x^2)^2}$$

and the generating function for $\{nL_n\}_{n\geq 0}$ is

$$xL'(x) = \frac{x+4x^2-x^3}{(-1+x+x^2)^2}.$$

Thus, the generating function for the sequence with general term $b_n = \frac{1}{10}\left[(5n-4)F_n - nL_n\right]$ is

$$B(x) = \frac{1}{10}\left[5\frac{x+x^3}{(-1+x+x^2)^2} - 4\frac{x}{1-x-x^2} - \frac{x+4x^2-x^3}{(-1+x+x^2)^2}\right] = \frac{x^3}{(-1+x+x^2)^2}.$$

Since $A(x) = B(x)$, the identity is proved. □

b) This identity is proved in an analogous way to the proof of a).

Let $A(x)$ be the generating function of the sequence

$$\left\{\sum_{k\geq 0} k^2\binom{n-k-1}{k}\right\}_{n\geq 0}.$$

Then,

$$\begin{aligned}A(x) &= \sum_{n\geq 0} x^n \sum_{k\geq 0} k^2\binom{n-k-1}{k} = \sum_{k\geq 0} k^2 x^{k+1} \sum_{n\geq 2k+1}\binom{n-k-1}{k} x^{n-k-1}\\ &= \sum_{k\geq 0} k^2 x^{k+1}\frac{x^k}{(1-x)^{k+1}} = \frac{x}{1-x}\sum_{k\geq 0} k^2\left(\frac{x^2}{1-x}\right)^k = \frac{x^3(-1+x-x^2)}{(-1+x+x^2)^3}.\end{aligned}$$

In the last step, we used $\sum_{k\geq 0} k^2 y^k = \dfrac{y+y^2}{(1-y)^3}$.

For the expression on the right-hand side of b), we obtain the same generating function by using that the generating function of $\left\{n^2F_n\right\}_{n\geq 0}$ is

$$x(xF'(x))' = -\frac{x(1+x+6x^2-x^3+x^4)}{(-1+x+x^2)^3}$$

and the generating function of $\left\{n^2L_n\right\}_{n\geq 0}$ is

$$x(xL'(x))' = \frac{x(-1-9x-9x^3+x^4)}{(-1+x+x^2)^3}.$$

□

## References

[1] OEIS Foundation Inc. (2011), The On-Line Encyclopedia of Integer Sequences, `http://oeis.org`.
[2] H. S. Wilf, *Generatingfunctionology*, Academic Press, Inc., Second ed. 1994.

**Also solved by Eduardo H. M. Brietzke, Paul S. Bruckman, Kenneth Davenport and the proposer.**

**Errata:** The correct answer to **H-691** in volume **49** no. 1, February 2012 should be

$$\sigma = \frac{G}{2} + \frac{\pi^2}{48} - \frac{7(\ln 2)^2}{8} - \frac{\pi \ln 2}{8};$$

i.e., the coefficient of $\pi^2$ should be 1/48 instead of 13/192. This is due to a missing factor of 1/4 in one integral.

The second problem labeled **H–723** in volume **49** no. 3, August 2012 should read **H–724**.