ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
FLORIAN LUCA

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, MATHEMATICAL INSTITUTE, UNAM, CP 04510, MEXICO DF, MEXICO or by e-mail at fluca@matmor.unam.mx as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-743 Proposed by Romeo Meštrović, Kotor, Montenegro.

Let $p \geq 5$ be a prime and $q_p(2) = (2^{p-1} - 1)/p$ be the Fermat quotient of p to base 2. Prove that

$$q_p(2) \equiv -\frac{1}{2} \sum_{k=1}^{(p-1)/2} \frac{(-3)^k}{k} \pmod{p}.$$

H-744 Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania.

Prove that

$$(1) \quad e^{n+3-L_{n+2}} \leq \left(\frac{1}{n} \sum_{k=1}^{n} \frac{1}{L_k} \right)^n; \qquad (2) \quad e^{n+2-L_n L_{n+1}} \leq \left(\frac{1}{n} \sum_{k=1}^{n} \frac{1}{L_k^2} \right)^n;$$

$$(3) \quad e^{n+1-F_{n+2}} \leq \left(\frac{1}{n} \sum_{k=1}^{n} \frac{1}{F_k} \right)^n; \qquad (4) \quad e^{n-F_n F_{n+1}} \leq \left(\frac{1}{n} \sum_{k=1}^{n} \frac{1}{F_k^2} \right)^n.$$

H-745 Proposed by Kenneth B. Davenport, SCI-Dallas, PA.

Prove that $(a^2 - 1)\cos(n+3)\theta - 2\sqrt{a}\cos n\theta = (a-1)^2 \cos(n+1)\theta$, where a is the real number satisfying $a^3 = a^2 + a + 1$ and θ is given by $\cos\theta = (1-a)\sqrt{a}/2$.

H-746 Proposed by H. Ohtsuka, Saitama, Japan.

Define the generalized Fibonomial coefficient $\binom{n}{k}_{F;m}$ by

$$\binom{n}{k}_{F;m} = \frac{F_{mn} F_{m(n-1)} \cdots F_{m(n-k+1)}}{F_{mk} F_{m(k-1)} \cdots F_m} \qquad \text{for} \quad 0 \leq k \leq n$$

with $\binom{n}{0}_{F;m} = 1$ and $\binom{n}{k}_{F;m} = 0$ (otherwise). Let $\varepsilon_i = (-1)^{(m+1)i}$. For positive integers n, m and s prove that

$$\sum_{i+j=2s} \varepsilon_i \binom{n}{i}_{F;m} \binom{n}{j}_{F;m} = \varepsilon_s \binom{n}{s}_{F;2m}.$$

SOLUTIONS

Fibonacci Numbers and Derivatives of Polynomials

H-717 Proposed by Samuel G. Moreno, Jaén, Spain, (Vol. 50, No. 2, May 2012)

Prove that if p is a polynomial such that $p(x) > 0$ for all $x \in \mathbb{R}$, then

$$\sum_{k=0}^{\deg(p)} F_{k+1} y^k p^{(k)}(x) > 0 \qquad \text{for all} \qquad x, y \in \mathbb{R}.$$

Solution by the proposer.

For a fixed $y \in \mathbb{R}$, $y \neq 0$, we consider the second-order linear differential equation with constant coefficients

$$(I - yD - y^2 D^2) q(x) = q(x) - y q'(x) - y^2 q''(x) = p(x), \tag{1}$$

in which I stands for the identity operator, and $D = d/dx$ stands for the derivative. If α denotes the golden ratio, the two distinct roots of the auxiliary equation of (1) are $\lambda_1 = -\alpha/y$ and $\lambda_2 = -(1-\alpha)/y$. Moreover, a particular solution of (1) is

$$\begin{aligned} q_0(x) &= \left(I - yD - y^2 D^2\right)^{-1} p(x) = \Big(\sum_{k=0}^{\infty} F_{k+1}\, y^k D^k\Big) p(x) \\ &= \sum_{k=0}^{\deg(p)} F_{k+1}\, y^k p^{(k)}(x). \end{aligned}$$

Thus, the general solution of (1) reads $q(x) = q_0(x) + C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$. Therefore, the unique polynomial solution of the differential equation considered is q_0.

Taking into account that p must be a polynomial of even degree, and also that the asymptotic behavior of q_0 is governed by $F_1 y^0 p^{(0)}(x) = p(x)$, we observe that q_0 tends to infinity as $|x|$ does, so there exists (at least) one absolute minimum m_0 of q_0 on $\mathbb{R}$. Using that $q_0'(m_0) = 0$ and $q_0''(m_0) \geq 0$, and using also (1), we conclude

$$q_0(x) \geq q_0(m_0) = p(m_0) + (y q_0'(m_0) + y^2 q_0''(m_0)) = p(m_0) + y^2 q_0''(m_0) > 0,$$

for all reals x.

Also solved by Paul S. Bruckman.

Inequalities with Fibonacci Numbers and Radicals

H-718 Proposed by Hideyuki Ohtsuka, Saitama, Japan.
(Vol. 50, No. 2, May 2012)

Let $A_{n,m} = F_{n+m}^{2n-2m-3}(F_{n+m}^4 - F_{n-m}^4)$. Prove that

(1) $\displaystyle\prod_{k=2m}^{2n} F_k \le A_{n,m}$ for $n \ge m \ge 1$;

(2) $\displaystyle\prod_{k=m}^{n} F_{2k} < \sqrt{A_{n,m}}\sqrt[4]{\frac{F_{2m-1}^3 F_{2n-1}F_{2n}}{F_{2m-3}F_{2m-2}F_{2n+1}}}$ for $n \ge m \ge 2$.

Solution by the proposer.

(1) Let $n \ge m \ge 1$. If $n = m$, then $LHS = RHS = F_{2n}$.
Let $n > m$. We have

$$\frac{\prod_{k=2m}^{2n} F_k}{F_{n+m}^{2n-2m+1}} = \prod_{k=2m}^{2n} \frac{F_k}{F_{n+m}} = \prod_{j=0}^{n-m} \frac{F_{n+m-j}}{F_{n+m}} \cdot \frac{F_{n+m+j}}{F_{n+m}}$$

$$= \prod_{j=0}^{n-m} \frac{F_{n+m}^2 - (-1)^{n+m-j}F_j^2}{F_{n+m}^2} \qquad \text{(By Catalan's Identity)}$$

$$= \prod_{j=0}^{n-m} \left(1 - \frac{(-1)^{n+m-j}F_j^2}{F_{n+m}^2}\right).$$

If $n - m$ is odd,

$$\prod_{j=0}^{n-m} \left(1 - \frac{(-1)^{n+m-j}F_j^2}{F_{n+m}^2}\right) = \prod_{r=0}^{(n-m-1)/2} \left(1 + \frac{F_{2r}^2}{F_{n+m}^2}\right)\left(1 - \frac{F_{2r+1}^2}{F_{n+m}^2}\right)$$

$$< \prod_{r=0}^{(n-m-1)/2} \left(1 + \frac{F_{2r+1}^2}{F_{n+m}^2}\right)\left(1 - \frac{F_{2r+1}^2}{F_{n+m}^2}\right) \le \prod_{r=0}^{(n-m-1)/2} \left(1 - \frac{F_{2r+1}^4}{F_{n+m}^4}\right)$$

$$\le 1 - \frac{F_{n-m}^4}{F_{n+m}^4}.$$

If $n - m$ is even,

$$\prod_{j=0}^{n-m} \left(1 - \frac{(-1)^{n+m-j}F_j^2}{F_{n+m}^2}\right) = \prod_{r=0}^{(n-m)/2} \left(1 + \frac{F_{2r-1}^2}{F_{n+m}^2}\right)\left(1 - \frac{F_{2r}^2}{F_{n+m}^2}\right)$$

$$\le \prod_{r=0}^{(n-m)/2} \left(1 + \frac{F_{2r}^2}{F_{n+m}^2}\right)\left(1 - \frac{F_{2r}^2}{F_{n+m}^2}\right) \le \prod_{r=0}^{(n-m)/2} \left(1 - \frac{F_{2r}^4}{F_{n+m}^4}\right)$$

$$\le 1 - \frac{F_{n-m}^4}{F_{n+m}^4}.$$

Therefore, we obtain

$$\prod_{k=2m}^{2n} F_k \leq F_{n+m}^{2n-2m+1}\left(1-\frac{F_{n-m}^4}{F_{n+m}^4}\right) = A_{n,m}.$$

(2) First, we have $F_{t-2}F_{t-1}F_{t+1}F_{t+2} < F_t^4$ by the Gelin–Cesàre Identity. Therefore, for $t \geq 3$, we have

$$\frac{F_{t-1}F_{t+1}}{F_t^2} < \frac{F_t^2}{F_{t-2}F_{t+2}}. \tag{1}$$

Let $n \geq m \geq 2$. We have

$$\prod_{k=m}^{n} \frac{F_{2k}^2}{F_{2k-1}^2} = \frac{F_{2n}}{F_{2m-2}} \prod_{k=m}^{n} \frac{F_{2k-2}F_{2k}}{F_{2k-1}^2} < \frac{F_{2n}}{F_{2m-2}} \prod_{k=m}^{n} \frac{F_{2k-1}^2}{F_{2k-3}F_{2k+1}} \quad \text{(by (1))}$$
$$= \frac{F_{2m-1}F_{2n-1}F_{2n}}{F_{2m-3}F_{2m-2}F_{2n+1}}.$$

Thus, we have

$$\prod_{k=m}^{n} F_{2k} < \sqrt{\frac{F_{2m-1}F_{2n-1}F_{2n}}{F_{2m-3}F_{2m-2}F_{2n+1}}} \prod_{k=m}^{n} F_{2k-1}.$$

Multiplying both sides of this inequality by $\prod_{k=m}^{n} F_{2k}$, we get

$$\prod_{k=m}^{n} F_{2k}^2 < \sqrt{\frac{F_{2m-1}F_{2n-1}F_{2n}}{F_{2m-3}F_{2m-2}F_{2n+1}}} \prod_{k=m}^{n} F_{2k-1}F_{2k}.$$

Here, we have

$$\prod_{k=m}^{n} F_{2k-1}F_{2k} = \prod_{k=2m-1}^{2n} F_k = F_{2m-1} \prod_{k=2m}^{2n} F_k \leq F_{2m-1}A_{n,m} \quad \text{(by (1))}.$$

Thus, we have

$$\prod_{k=m}^{n} F_{2k}^2 < A_{n,m}\sqrt{\frac{F_{2m-1}^3F_{2n-1}F_{2n}}{F_{2m-3}F_{2m-2}F_{2n+1}}},$$

which leads to the desired inequality.

Note. We obtain the following inequality in the same manner as (2):

$$\prod_{k=m}^{n} F_{2k-1} < \sqrt{A_{n,m}}\sqrt[4]{\frac{F_{2m-1}^3F_{2n-1}F_{2n}}{F_{2m-2}F_{2n+1}F_{2n+2}}} \qquad \text{(for } n \geq m \geq 2\text{)}.$$

Also solved by Paul S. Bruckman and Dmitry Fleischman.

Alternating Sums of High Powers of Fibonacci Numbers

H-719 Proposed by Hideyuki Ohtsuka, Saitama, Japan.
(Vol. 50, No. 2, May 2012)

Let $T_j(n) = (-1)^{n(j+1)}(F_nF_{n+1})^j$. Given a positive integer m prove that there are rational numbers $\lambda_1, \ldots, \lambda_m$ such that

$$\sum_{k=1}^{n}(-1)^{k(m+1)}F_k^{2m} = \sum_{j=1}^{m}\lambda_j T_j(n).$$

Show the identities

(1) $\displaystyle\sum_{k=1}^{n}(-1)^k F_k^4 = -\frac{2}{3}T_1(n) + \frac{1}{3}T_2(n);$

(2) $\displaystyle\sum_{k=1}^{n} F_k^6 = \frac{1}{2}T_1(n) - \frac{1}{4}T_2(n) + \frac{1}{4}T_3(n);$

(3) $\displaystyle\sum_{k=1}^{n}(-1)^k F_k^8 = -\frac{8}{21}T_1(n) + \frac{4}{21}T_2(n) - \frac{2}{7}T_3(n) + \frac{1}{7}T_4(n).$

Solution by Harris Kwong, SUNY Fredonia, NY.

Lemma. *For any integer $i \geq 1$, there exist rational numbers $a_{i,\ell}$ such that*

$$F_k^i = F_{k+1}^i + (-1)^i F_{k-1}^i + \sum_{\ell=0}^{\lfloor i/2 \rfloor} a_{i,\ell}(-1)^{k\ell}F_k^{i-2\ell}.$$

Equivalently, we can write

$$F_{k+1}^i + (-1)^i F_{k-1}^i = \sum_{\ell=0}^{\lfloor i/2 \rfloor} b_{i,\ell}(-1)^{k\ell}F_k^{i-2\ell}$$

for some rational numbers $b_{i,\ell}$.

Proof. Induct on i. The result is obviously true when $i = 1$, because $F_{k+1} - F_{k-1} = F_k$. For $i \geq 2$,

$$F_k^i = (F_{k+1} - F_{k-1})^i = F_{k+1}^i + (-1)^i F_{k-1}^i + \sum_{r=1}^{i-1}(-1)^r\binom{i}{r}F_{k+1}^{i-r}F_{k-1}^r.$$

When i is even, Casini's identity $F_{k+1}F_{k-1} = F_k^2 + (-1)^k$ implies that the middle term in the summation, where $r = i/2$, is

$$\begin{aligned}(-1)^{i/2}\binom{i}{i/2}(F_{k+1}F_{k-1})^{i/2} &= (-1)^{i/2}\binom{i}{i/2}[F_k^2 + (-1)^k]^{i/2}\\ &= (-1)^{i/2}\binom{i}{i/2}\sum_{\ell=0}^{i/2}\binom{i/2}{\ell}F_k^{2(i/2-\ell)}(-1)^{k\ell}\\ &= \sum_{\ell=0}^{i/2} c_{i/2,\ell}(-1)^{k\ell}F_k^{i-2\ell},\end{aligned}$$

where $c_{i/2,\ell} = (-1)^{i/2}\binom{i}{i/2}\binom{i/2}{\ell}$.

In general, for $1 \le r \le \lfloor (i-1)/2 \rfloor$, due to symmetry, we can group the rth term with the $(i-r)$th term; and it follows from the induction hypothesis that

$$\begin{aligned}
&(-1)^r\binom{i}{r}F_{k+1}^{i-r}F_{k-1}^{r} + (-1)^{i-r}\binom{i}{i-r}F_{k+1}^{r}F_{k-1}^{i-r}\\
&= (-1)^r\binom{i}{r}(F_{k+1}F_{k-1})^r[F_{k+1}^{i-2r} + (-1)^{i-2r}F_{k-1}^{i-2r}]\\
&= (-1)^r\binom{i}{r}(F_k^2 + (-1)^k]^r[F_{k+1}^{i-2r} + (-1)^{i-2r}F_{k-1}^{i-2r}]\\
&= (-1)^r\binom{i}{r}\left[\sum_{s=0}^{r}\binom{r}{s}F_k^{2(r-s)}(-1)^{ks}\right]\left[\sum_{t=0}^{\lfloor (i-2r)/2\rfloor} b_{i-2r,t}(-1)^{kt}F_k^{i-2r-2t}\right]\\
&= \sum_{\ell=0}^{\lfloor i/2\rfloor} c_{r,\ell}(-1)^{k\ell}F_k^{i-2\ell},
\end{aligned}$$

where $c_{r,\ell} = \sum_{s+t=\ell}(-1)^r\binom{i}{r}rsb_{i-2r,t}$ is a rational number. The result follows immediately. □

We now prove the original problem. The case of $m = 1$ is valid:

$$\sum_{k=1}^{n} F_k^2 = F_nF_{n+1} = T_1(n).$$

Since $F_k^{2m} = F_k^m \cdot F_k^m$, the lemma asserts that

$$\begin{aligned}
\sum_{k=1}^{n}(-1)^{k(m+1)}F_k^{2m} &= \sum_{k=1}^{n}(-1)^{k(m+1)}F_k^{m}\left(F_{k+1}^m + (-1)^m F_{k-1}^m + \sum_{\ell=0}^{\lfloor m/2\rfloor} a_{m,\ell}(-1)^{k\ell}F_k^{m-2\ell}\right)\\
&= (-1)^{n(m+1)}F_n^mF_{n+1}^m + \sum_{\ell=0}^{\lfloor m/2\rfloor} a_{m,\ell}\sum_{k=1}^{n}(-1)^{k(m-\ell+1)}F_k^{2(m-\ell)}.
\end{aligned}$$

Solving for $\sum_{k=1}^{n}(-1)^{k(m+1)}F_k^{2m}$ yields the desired result from induction.

In practice, it is easier to compute the coefficients λ_j directly. For example, when $m = 2$,

$$F_k^2 = (F_{k+1} - F_{k-1})^2 = F_{k+1}^2 + F_{k-1}^2 - 2F_{k+1}F_{k-1} = F_{k+1}^2 + F_{k-1}^2 - 2F_k^2 - 2(-1)^k.$$

This leads to

$$\begin{aligned}
\sum_{k=1}^{n}(-1)^kF_k^4 &= \sum_{k=1}^{n}(-1)^kF_k^2(F_{k+1}^2 + F_{k-1}^2) - 2\sum_{k=1}^{n}(-1)^kF_k^4 - 2\sum_{k=1}^{n}F_k^2\\
&= (-1)^nF_n^2F_{n+1}^2 - 2\sum_{k=1}^{n}(-1)^kF_k^4 - 2T_1(n).
\end{aligned}$$

Thus, $3\sum_{k=1}^{n}(-1)^kF_k^4 = T_2(n) - 2T_1(n)$, which proves (1).

In a similar manner, we find

$$F_k^3 = F_{k+1}^3 - F_{k-1}^3 - 3F_{k+1}F_{k-1}(F_{k+1} - F_{k-1}) = F_{k+1}^3 - F_{k-1}^3 - 3[F_k^2 + (-1)^k]F_k.$$

Hence,

$$\sum_{k=1}^{n} F_k^6 = \sum_{k=1}^{n} F_k^3(F_{k+1}^3 - F_{k-1}^3) - 3\sum_{k=1}^{n} F_k^6 - 3\sum_{k=1}^{n}(-1)^k F_k^4$$
$$= F_n^3 F_{n+1}^3 - 3\sum_{k=1}^{n} F_k^6 - 3\left(\frac{1}{3}\,T_2(n) - \frac{2}{3}\,T_1(n)\right).$$

This yields $4\sum_{k=1}^{n} F_k^6 = T_3(n) - T_2(n) + 2T_1(n)$, thereby proving (2).

The case of $m = 4$ is slightly more complicated. First we obtain

$$\begin{aligned}
F_k^4 &= F_{k+1}^4 + F_{k-1}^4 - 4F_{k+1}F_{k-1}(F_{k+1}^2 + F_{k-1}^2) + 6F_{k+1}^2F_{k-1}^2 \\
&= F_{k+1}^4 + F_{k-1}^4 - 4F_{k+1}F_{k-1}[(F_{k+1} - F_{k-1})^2 + 2F_{k+1}F_{k-1}] + 6F_{k+1}^2F_{k-1}^2 \\
&= F_{k+1}^4 + F_{k-1}^4 - 4F_{k+1}F_{k-1}F_k^2 - 2F_{k+1}^2F_{k-1}^2 \\
&= F_{k+1}^4 + F_{k-1}^4 - 4[F_k^2 + (-1)^k]F_k^2 - 2[F_k^2 + (-1)^k]^2 \\
&= F_{k+1}^4 + F_{k-1}^4 - 6F_k^4 - 8(-1)^kF_k^2 - 2.
\end{aligned}$$

Therefore,

$$\sum_{k=1}^{n}(-1)^k F_k^8 = \sum_{k=1}^{n}(-1)^k F_k^4(F_{k+1}^4 + F_{k-1}^2) - 6\sum_{k=1}^{n}(-1)^k F_k^8 - 8\sum_{k=1}^{n} F_k^6 - 2\sum_{k=1}^{n}(-1)^k F_k^4.$$

We conclude that

$$\sum_{k=1}^{n}(-1)^n F_k^8 = \frac{1}{7}\left[T_4(n) - 8\left(\frac{1}{4}\,T_3(n) - \frac{1}{4}\,T_2(n) + \frac{1}{2}\,T_1(n)\right) - 2\left(\frac{1}{3}\,T_2(n) - \frac{2}{3}\,T_1(n)\right)\right]$$
$$= \frac{1}{7}\,T_4(n) + \frac{2}{7}\,T_3(n) + \frac{4}{21}\,T_2(n) - \frac{8}{21}\,T_1(n),$$

which establishes (3).

Also solved by Paul S. Bruckman, Kenneth B. Davenport, Dmitry Fleischman and Zbigniew Jakubczyk.

Late Acknowledgements. Kenneth B. Davenport, M. N. Deshpande, Harris Kwong, and Anastasios Kotronis all solved **H-716**.