ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Russ Euler and Jawad Sadek

Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by August 15, 2005. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2,\ \beta = (1-\sqrt{5})/2,\ F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-991 Proposed by Peter Jeuck, Hewitt, NJ

Consider a 3×3 magic square of the following form.

F_n	F_{n+2}	F_{n+3}
b_1	b_2	b_3
c_1	c_2	c_3

Prove or disprove: The integer b_1 must be a Lucas number.

B-992 Proposed by the Problem Editor

Prove or disprove: $L_{6n}^2 \equiv 4 \pmod{10}$ for all integers n.

B-993 Proposed by Miquel Grau and José Luis Díaz-Barrero, Universitat Politécnica de Catalunya, Barcelona, Spain

Let n be a nonnegative integer. Prove that

$$F_n^2(F_n - 2F_{n+1} - F_{2n}) + F_{2n}(F_n + 2F_nF_{n+1} - F_{2n}) = 0,$$

$$L_n^2(L_n - 2F_{n+1} - F_{2n}) + F_{2n}(L_n + 2L_nF_{n+1} - F_{2n}) = 0.$$

B-994 Proposed by Juan Pla, Paris, France

Under what condition(s) on k, if any, does $L_k + 2$ divide $\frac{F_{kn}}{F_k} + (-1)^n n$ for all integers $n \geq 0$?

B-995 Proposed by Mario Catalani, University of Torino, Torino, Italy

Let $L_n \equiv L_n(x,y)$ be the Lucas polynomials defined by $L_0 = 2, L_1 = x$ and for $n \geq 2,\ L_n = L_{n-1} + L_{n-2}$. Assume $x \neq 0,\ y \neq 0$, and $x^2 + 4y \neq 0$. Prove the following identities $(n \geq 1)$:

1. $$\sum_{k=1}^{n} \binom{n}{k} kL_{k-1}(x,y) = nL_{n-1}(x+2, y-x-1),$$

2. $$\sum_{k=1}^{n} \binom{n}{k} kx^{k-1}y^{n-k}L_k(x,y) = nL_{2n-1}(x,y).$$

SOLUTIONS

A Pythagorean-Like Equality

B-976 Proposed by Muneer Jebreel Karameh, Jerusalem, Israel (Vol. 42, no. 2, May 2004)

Prove that $(L_nL_{n+3})^2 + (2L_{n+1}L_{n+2})^2 = (L_{2n+5} - L_{2n+1})^2$.

Solution I by Charles K. Cook, University of South Carolina Sumter, Sumter, SC

In [2] Umansky and Tallman proposed an identity solved by Swamy in [1]. Namely,

$$(L_nL_{n+3})^2 + (2L_{n+1}L_{n+2})^2 = (L_{n+2}L_{n+3} - L_nL_{n+1})^2.$$

So it follows that

$$\begin{aligned}(L_{n+2}L_{n+3} - L_nL_{n+1})^2 &= \left[\left(\alpha^{n+2} + \beta^{n+2}\right)\left(\alpha^{n+3} + \beta^{n+3}\right) - (\alpha^n + \beta^n)\left(\alpha^{n+1} + \beta^{n+1}\right)\right]^2 \\ &= \left[\alpha^{2n+5} + (-1)^n(\alpha+\beta) + \beta^{2n+5} - \alpha^{2n+1}\right. \\ &\qquad \left. -(-1)^n(\alpha+\beta) - \beta^{2n+1}\right]^2 \\ &= (L_{2n+5} - L_{2n+1})^2.\end{aligned}$$

Note: The above identity of Umansky and Tallman as given in [3 page 91 #89] has an error: $(2L_{n+1}L_{n+2})^2$ appears as $(2L_{n+1}L_{n+3})^2$.

References:

1. M.N.S. Swamy. "Pythagoreans and All that Stuff." *The Fibonacci Quarterly* **6.4** (1968): 259.
2. Harlan Umansky and Malcolm Tallman. "Problem H-101." *The Fibonacci Quarterly* **4.4** (1966): 333.
3. Thomas Koshy. *Fibonacci and Lucas Numbers with Applications.* New York, John Wiley & Sons, Inc., 2001.

Solution II by Paul S. Bruckman, Sointula, BC V0N 3E0, Canada

Let L and R denote the left and right side, respectively, of the putative identity.

Note that $R = (L_{2n+4} + L_{2n+2})^2 = (5F_{2n+3})^2$. We employ the following identities which can be established by means of Binet's formulas.

$$L_u L_v = L_{u+v} + (-1)^v L_{u-v} \tag{1}$$

$$(L_u)^2 - 5(F_u)^2 = 4(-1)^u. \tag{2}$$

Thus, $L = \{L_{2n+3} + 4(-1)^n\}^2 + \{2L_{2n+3} - 2(-1)^n\}^2 = 5(L_{2n+3})^2 + 20 = (5F_{2n+3})^2 = R$.

Also solved by Gurdial Arora, Steve Edwards (two solutions), G.C. Greubel, Ovidiu Furdui, Russell Hendel, Pentti Haukkanen, Walther Janous, Harris Kwong, Jaroslav Seibert (two solutions), H.-J. Seiffert, James Sellers, and the proposer.

The Integral Connection

B-977 Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria (Vol. 42, no. 2, May 2004)

Determine all integers n such that $\alpha^n = a + b\sqrt{5}$ where a and b are integers.

Solution by James A. Sellers, Department of Mathematics, Pennsylvania State University, University Park, PA

We first note that

$$\alpha^n = \frac{L_n + F_n\sqrt{5}}{2}.$$

This is easily seen from $F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)$ and $L_n = \alpha^n + \beta^n$.

Thus, in order to satisfy the requirements of the problem, we need $\frac{L_n}{2}$ and $\frac{F_n}{2}$ to be integers. That means we must find values n such that both L_n and F_n are even. This is true precisely when n is a multiple of three.

All solutions received were more or less the same as the featured solution. Also solved by Gurdial Arora, Paul S. Bruckman, Mario Catalani, Charles K. Cook, Steve Edwards, Ovidiu Furdui, G.C. Greubel, Russell Hendel, Pentti Haukkanen,

Walther Janous, Harris Kwong, Carl Libis, Jaroslav Seibert, H.-J. Seiffert, James Sellers, and the proposer.

Determine the Determinant

B-978 Proposed by Carl Libis, University of Rhode Island, Kingston, RI (Vol. 42, no. 2, May 2004)

For $n > 0$, let $A_n = [a_{i,j}]$ denote the symmetric matrix with $a_{i,i} = i+1$ and $a_{i,j} = \min\{i, j\}$ for all integers i and j with $i \neq j$. Find the determinant of A_n.

Solution by Jaroslav Seibert, University Hradec Kralove, The Czech Republic

For any integer $n > 0$ the determinant of A_n is equal to F_{2n+1}. The proof is by induction on n.

It is easy to see that

$$\det A_1 = |2| = 2 = F_3$$

$$\det A_2 = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 5 = F_5.$$

Suppose now that the statement is true for any integers $n-1$ and $n-2$. For an integer n

$$\det A_n = \begin{vmatrix} 2 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 3 & 2 & \cdots & 2 & 2 \\ 1 & 2 & 4 & \cdots & 3 & 3 \\ & & . & \cdots & . & \\ 1 & 2 & 3 & \cdots & n & n-1 \\ 1 & 2 & 3 & \cdots & n-1 & n+1 \end{vmatrix}.$$

Subtract the $(n-1)^{st}$ column from the n^{th} column and then subtract the $(n-1)^{st}$ row from the n^{th} row to get

$$\det A_n = \begin{vmatrix} 2 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 3 & 2 & \cdots & 2 & 0 \\ 1 & 2 & 4 & \cdots & 3 & 0 \\ & & . & \cdots & . & \\ 1 & 2 & 3 & \cdots & n & -1 \\ 1 & 2 & 3 & \cdots & n-1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 3 & 2 & \cdots & 2 & 0 \\ 1 & 2 & 4 & \cdots & 3 & 0 \\ & & . & \cdots & . & \\ 1 & 2 & 3 & \cdots & n & -1 \\ 0 & 0 & 0 & \cdots & -1 & 3 \end{vmatrix}$$

Expanding with respect to the last row gives

$$\det A_n = 3\det A_{n-1} + \begin{vmatrix} 2 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 3 & 2 & \cdots & 2 & 2 \\ 1 & 2 & 4 & \cdots & 3 & 3 \\ & & . & \cdots & . & \\ 1 & 2 & 3 & \cdots & n-1 & n-2 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{vmatrix} = 3\det A_{n-1} - \det A_{n-2}.$$

It follows that

$$\det A_n = 3F_{2n-1} - F_{2n-3} = 2F_{2n-1} + F_{2n-2} = F_{2n} + F_{2n-1} = F_{2n+1},$$

and the proof is complete.

Mario Catalani proved the result taking the product of the eigenvalues of A_n.

Also solved by Paul S. Bruckman, Mario Catalani, Steve Edwards, Russell Hendel, Harris Kwong, Walther Janous, Kathleen Lewis, and the proposer. One incomplete solution was received.

The Limit Vanishes

B-979 Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, MI (Vol. 42, no. 2, May 2004)

Prove that

$$\lim_{n\to\infty}\left[n\left(\sqrt[n+1]{F_{n+1}} - \sqrt[n]{F_n}\right)\right] = 0.$$

Solution by Harris Kwong, SUNY Fredonia, Fredonia, NY

We shall prove a more general result. First note that for any nonnegative integer ℓ,

$$F_{n+\ell} = \frac{\alpha^{n+\ell} - \beta^{n+\ell}}{\alpha - \beta} = \frac{\alpha^{n+\ell}}{\alpha - \beta}\left[1 - \left(\frac{\beta}{\alpha}\right)^{n+\ell}\right] \sim \frac{\alpha^{n+\ell}}{\alpha - \beta}$$

and

$$n\binom{\frac{1}{n+\ell}}{k} = \frac{n}{k!}\cdot\frac{1}{n+\ell}\left(\frac{1}{n+\ell} - 1\right)\left(\frac{1}{n+\ell} - 2\right)\cdots\left(\frac{1}{n+\ell} - k + 1\right) \sim \frac{(-1)^{k-1}}{k}.$$

It follows that if we write $1/(\alpha - \beta) = 1 - t$, then $0 < t < 1$, and

$$\begin{aligned} n\left(\sqrt[n+m]{F_{n+m}} - \sqrt[n]{F_n}\right) &\sim n\alpha\left[(1-t)^{\frac{1}{n+m}} - (1-t)^{\frac{1}{n}}\right] \\ &= \alpha\sum_{k=0}^{\infty} n\left[\binom{\frac{1}{n+m}}{k} - \binom{\frac{1}{n}}{k}\right](-t)^n \\ &\sim 0 \end{aligned}$$

for all nonnegative integers m.

Also solved by Paul S. Bruckman, G.C. Greubel, Russell Hendel, Walther Janous, and the proposer.

Always One

B-980 Proposed by Steve Edwards, Southern Polytechnic State University, Marietta, GA
(Vol. 42, no. 2, May 2004)

For any integers m and n, evaluate

$$\left(L_n\alpha^m + L_{n-1}\alpha^{m-1}\right) / \left(L_m\alpha^n + L_{m-1}\alpha^{n-1}\right).$$

Solution by Kathleen E. Lewis, Department of Mathematics, SUNY Oswego, Oswego, NY

The numerator of the fraction is equal to

$$\begin{aligned}(\alpha^n + \beta^n)\alpha^m &+ (\alpha^{n-1} + \beta^{n-1})\alpha^{m-1} \\ &= \alpha^{n+m} + \alpha^m\beta^n + \alpha^{n+m-2} + \alpha^{m-1}\beta^{n-1} \\ &= \alpha^{n+m-2}[\alpha^2 + 1] + \alpha^{m-1}\beta^{n-1}[\alpha\beta + 1].\end{aligned}$$

But $\alpha\beta = -1$, so the second half of the last formula vanishes, leaving a numerator of $\alpha^{n+m-2}[\alpha^2 + 1]$. In the same way, the denominator can be written as

$$\begin{aligned}(\alpha^m + \beta^m)\alpha^n &+ (\alpha^{m-1} + \beta^{m-1})\alpha^{n-1} \\ &= \alpha^{m+n-2}[\alpha^2 + 1] + \alpha^{n-1}\beta^{m-1}[\alpha\beta + 1] \\ &= \alpha^{m+n-2}[\alpha^2 + 1].\end{aligned}$$

Therefore, the numerator and denominator are equal, so the fraction is equal to 1.

All solutions were similar to the featured one. Also solved by Gurdial Arora, Paul S. Bruckman, Mario Catalani, Charles K. Cook, G.C. Greubal, Pentti Haukkanen, Russell Hendel, Walther Janous, Harris Kwong, Jaroslav Seibert, H. -J. Seiffert, James Sellers, and the proposer.