FIBONACCI EXPONENTIALS AND GENERALIZATIONS OF HERMITE POLYNOMIALS

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Little seems to be known about series of the form

$$\sum_{n=0}^{\infty} A_n x^{F_n} \tag{1}$$

or

$$\sum_{n=0}^{\infty} A_n x^{L_n} \quad , \tag{2}$$

where the exponents are Fibonacci and Lucas numbers, respectively, defined by

$$F_n = \frac{a^n - b^n}{a - b} \,, \quad L_n = a^n + b^n \,, \quad a \neq b \quad . \tag{3}$$

It may therefore be of interest to point out that Fibonacci exponentials are intimately related to some generalizations of Hermite polynomials [1]. The existence (or non-existence) of certain generating functions for these generalized Hermite polynomials would possibly shed some light on series of the type (1) and (2).

In the paper [1], a function $H_n^r(x, a, p)$ was introduced by the definition

$$H_n^r(x, a, p) = (-1)^n x^{-a} e^{px^r} D_x^n (x^a e^{-px^r}) \quad , \tag{4}$$

which gave the generating function

$$\left(1 - \frac{t}{x}\right)^a e^{p(x^r - (x-t)^r)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n^r(x, a, p) \quad . \tag{5}$$

This expansion gives at once in a formal sense

$$x^{F_n} = e^{p(a-b)F_n} = \left(\frac{a}{b}\right)^m \sum_{k=0}^{\infty} \frac{(a - b)^k}{k!} H_k^n(a, m, p) \quad , \tag{6}$$

where p, x satisfy $p(a - b) = \log x$.

Therefore we have

$$\sum_{n=0}^{\infty} A_n t^n x^{F_n} = \left(\frac{a}{b}\right)^m \sum_{k=0}^{\infty} \frac{(a-b)^k}{k!} \sum_{n=0}^{\infty} A_n t^n H_k^n(a,m,p) \quad , \tag{7}$$

from which it is evident that it would be desirable to establish simple generating functions of the sort

$$G_1 = \sum_{n=0}^{\infty} A_n t^n H_k^n(a,m,p) \tag{8}$$

for the generalized Hermite polynomials.

For the Lucas numbers we have

$$x^{L_n} = x^{a^n} \cdot x^{b^n} = e^{pa^n} \cdot e^{pb^n} \quad , \quad \text{with } p = \log x \ ,$$

and, formally, we have from (5)

$$e^{pa^n} = \sum_{k=0}^{\infty} \frac{a^k}{k!} H_k^n(a,0,p) \quad . \tag{9}$$

Consequently we find

$$x^{L_n} = \sum_{k=0}^{\infty} \frac{b^k}{k!} \sum_{j=0}^{k} \binom{k}{j} \left(\frac{a}{b}\right)^j H_j^n(a,0,p) H_{k-j}^n(b,0,p) \quad . \tag{10}$$

With this approach to a series of the type (2) we should next have to find bilinear generating functions of the form

$$G_2 = \sum_{n=0}^{\infty} A_n t^n H_j^n(a,u,p) H_k^n(b,v,p) \quad , \tag{11}$$

which seem difficult to obtain. Of course this is not the only way to relate the Lucas numbers to the H functions, but it is suggestive of new avenues of research.

One may readily verify (as was found in [1]) that an explicit formula for the H functions is

$$H_n^r(x,a,p) = (-1)^n n! \sum_{k=0}^{n} p^k \frac{x^{rk-n}}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{a+rj}{n} \quad . \tag{12}$$

In (7) m is a parameter and we may take m = 0 for our purposes. Thus we find

$$\sum_{n=0}^{\infty} A_n t^n H_k^n(a,0,p)$$

(13)

$$= (-1)^k k!\, a^{-k} \sum_{s=0}^{k} \frac{p^s}{s!} \sum_{j=0}^{s} (-1)^j \binom{s}{j} \sum_{n=0}^{\infty} A_n \binom{nj}{k} (ta^s)^n ,$$

so that we should have to find some really <u>simple</u> sum for a series of the type

$$(14) \qquad \sum_{n=0}^{\infty} A_n \binom{nj}{k} z^n ,$$

and this also seems difficult. In the case $A_n = 1$ (for all n) it <u>is</u> possible to sum this series as follows.

In general

$$(15) \qquad \sum_{n=0}^{m} f(jn) = \frac{1}{j} \sum_{s=1}^{j} \sum_{n=0}^{jm} \omega_j^{sn} f(n), \quad \text{with } \omega_j = e^{2\pi i/j} .$$

This gives the summation formula

$$(16) \qquad \sum_{n=0}^{\infty} \binom{jn}{k} t^{jn} = \frac{1}{j} \sum_{s=1}^{j} \frac{(t\omega_j^s)^k}{(1 - t\omega_j^s)^{k+1}} , \qquad |t| < 1, \; j \geq 1 ,$$

so that in principle we have a (complicated) generating function for (13).

Another direction in which we may go to find generating functions is suggested by the second generalization of Hermite polynomials given in [1]. By definition

$$(17) \qquad g_n^r(x,h) = e^{hD^r} x^n , \quad D = D_x ,$$

and this yields the generating function

$$(18) \qquad e^{tx+ht^r} = \sum_{n=0}^{\infty} \frac{t^n}{n!} g_n^r(x,h) .$$

Thus in a formal sense

$$e^{pa^n} = e^{-az} \sum_{k=0}^{\infty} \frac{a^k}{k!} g_k^n (z,p) \quad . \tag{19}$$

Two such expansions, with parameters a and b, might be multiplied together or perhaps combined with the expansion (9) in order to obtain generating functions involving Fibonacci and Lucas numbers as exponents. It seems clear that what is needed is a collection of interesting and simple generating functions for the generalized Hermite polynomials. It is hoped to offer further results in this direction in a later paper.

REFERENCE

1. H. W. Gould and A. T. Hopper, Operational Formulas Connected With Two Generalizations of Hermite Polynomials, Duke Mathematical Journal, 29 (1962), 51 — 63.

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1. J. A. Jeske, Linear Recurrence Relations — Part I, the Fibonacci Quarterly, Vol. 1, No. 2, pp. 69—74.
2. G. N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge, 1944.

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