8. AN EXTENDED RESULT

Theorem 5. The series
\[ A = \sum_{m=1}^{\infty} (-1)^{m+1} \tan^{-1} \frac{1}{F_{2m}} \]
converges and \( A = \tan^{-1} \left( \frac{\sqrt{5} - 1}{2} \right) \).

Proof: Since the series is an alternating series, and, since \( \tan^{-1} x \) is a continuous increasing function, then
\[ \tan^{-1} \frac{1}{F_{2n}} > \tan^{-1} \frac{1}{F_{2n+2}} \quad \text{and} \quad \tan^{-1} 0 = 0. \]

The angle \( A \) must lie between the partial sums \( S_N \) and \( S_{N+1} \) for every \( N > 2 \) by the error bound in the alternating series, but \( S_N = \tan^{-1} \left( \frac{F_N}{F_{N+1}} \right) \). Thus the angles of \( U_N \) and \( U_{N+1} \) lie on opposite sides of \( A \). By the continuity of \( \tan^{-1} x \) then
\[ \lim_{n \to \infty} \tan^{-1} \left( \frac{F_n}{F_{n+1}} \right) = A = \tan^{-1} \left( \frac{\sqrt{5} - 1}{2} \right). \]

Comment: The same result can be obtained simply from
\[ \tan \left( \tan^{-1} \frac{F_n}{F_{n+1}} - \frac{\sqrt{5} - 1}{2} \right) = (-1)^{n+1} \left( \frac{\sqrt{5} - 1}{2} \right)^{2n+1} \]
Which slope gives a better numerical approximation to \( \frac{\sqrt{5} - 1}{2} \), \( F_n/F_{n+1} \) or \( L_n/L_{n+1} \)? Hmmm?

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