A GENERATING FUNCTION FOR PARTLY ORDERED PARTITIONS

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1. In a recent paper [1], Cadogan has discussed the function $\phi_k(n)$ which satisfies the recurrence

$$\phi_k(n) = \phi_k(n-1) + \phi_{k-1}(n-1) \qquad (n > k \ge 1) \tag{1}$$

together with

$$\phi_0(n) = p(n) \tag{2}$$

and

$$\phi_k(k) = 2^{k-1} \qquad (k \ge 1) . \tag{3}$$

As usual $p(n)$ denotes the number of unrestricted partitions of n, so that

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{n=1}^{\infty} (1 - x^n)^{-1} . \tag{4}$$

The object of the present note is to obtain a generating function for $\phi_k(n)$. Put

$$\Phi_k(x) = \sum_{n=k}^{\infty} \phi_k(n)\, x^n ,$$

$$\Phi(x,y) = \sum_{k=0}^{\infty} \Phi_k(x)\, y^k = \sum_{n,k=0}^{\infty} \phi_k(n)\, x^n y^k .$$

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Then, by (1) and (3), we have

$$\Phi_k(x) = 2^{k-1}x^k + \sum_{n=k+1}^{\infty}\{\phi_k(n-1) + \phi_{k-1}(n-1)\}x^n$$

$$= 2^{k-1}x^k + x\sum_{n=k}^{\infty}\phi_k(x^n)x^n + x\sum_{n=k}^{\infty}\phi_{k-1}(n)\,x^n$$

$$= 2^{k-1}x^k + x\Phi_k(x) + x\Phi_{k-1}(x) - \phi_{k-1}(k-1)x^k \;,$$

so that

$$(1 - x)\Phi_1(x) = x\Phi_0(x) \;, \tag{5}$$

$$(1 - x)\Phi_k(x) = 2^{k-2}x^k = x\Phi_{k-1}(x) \qquad (k > 1) \,. \tag{6}$$

It follows that

$$\Phi(x,y) = \Phi_0(x) + \Phi_1(x)y + \sum_{k=2}^{\infty}\Phi_k(x)\,y^k$$

$$= \Phi_0(x) + \frac{xy}{1-x}\Phi_0(x) + \frac{1}{1-x}\sum_{k=2}^{\infty}\{2^{k-2}x^k + x\Phi_{k-1}(x)\}\,y^k$$

$$= \Phi_0(x) + \frac{x^2y^2}{(1-x)(1-xy)} + \frac{xy}{1-x}\Phi(x,y) \;.$$

We have therefore

$$\begin{aligned}\Phi(x,y) &= \frac{(1-x)\Phi_0(x)}{1-x-xy} + \frac{x^2y^2}{(1-x-xy)(1-2xy)} \\ &= \frac{1-x}{1-x-xy}\prod_{n=1}^{\infty}(1-x^n)^{-1} + \frac{x^2y^2}{(1-x-xy)(1-2xy)} \;.\end{aligned} \tag{7}$$

2. By means of (7) we can obtain an explicit formula for $\Phi_k(x)$. Since

$$\frac{1-x}{1-x-xy} = \left(1 - \frac{xy}{1-x}\right)^{-1} = \sum_{k=0}^{\infty} \frac{x^k y^k}{(1-x)^k}$$

and

$$\frac{1}{(1-x-xy)(1-2xy)} = \sum_{r=0}^{\infty} \frac{x^r y^r}{(1-x)^{r+1}} \sum_{s=0}^{\infty} (2xy)^s$$

$$= \sum_{k=0}^{\infty} x^k y^k \sum_{r=0}^{k} \frac{2^{k-r}}{(1-x)^{r+1}} ,$$

it follows that

$$\Phi_k(x) = \frac{x^k}{(1-x)^k}\,\Phi_0(x) + \sum_{r=0}^{k-2} \frac{2^{k-r-2}\,x^k}{(1-x)^{r+1}} . \tag{8}$$

Moreover, since

$$\frac{1}{(1-x)^{r+1}} = \sum_{s=0}^{\infty} \binom{r+s}{r} x^s ,$$

Eq. (8) implies

$$\phi_k(n) = \sum_{r=0}^{n-k} \binom{k+r-1}{r} p(n-k-r) + \sum_{r=0}^{k-2} 2^{k-r-2} \binom{n-k+r}{r} \qquad (k \geq 2) \tag{9}$$

For $k = 1$, we have

$$\phi_1(n) = \sum_{r=0}^{n-1} p(n-r) \tag{10}$$

as is evident from (5).

Replacing k by n - k in (9) we get

$$\phi_{n-k}(n) = \sum_{r=0}^{k} \binom{n-k+r-1}{r} p(k-r) + \sum_{r=0}^{n-k-2} 2^{n-k-r-2}\binom{k+r}{r} . \qquad (n \geq k+2) \tag{11}$$

Cadogan [1] has derived the formula

$$\begin{aligned} \phi_{n-k}(n) &= \sum_{r=3}^{k} \binom{n-r-1}{k-r} p(r) + \sum_{r=0}^{n-k-1} \binom{k+r-3}{r} 2^{n-k-r+1} \\ &= \sum_{r=0}^{k-3} \binom{n-k+r-1}{r} p(k-r) + \sum_{r=0}^{n-k-1} \binom{k+r-3}{r} 2^{n-k-r+1} . \qquad (3 \leq k < n,\ n \geq 4) \end{aligned} \tag{12}$$

To show that (11) and (12) are in agreement, it suffices to verify that

$$\begin{aligned} &\sum_{r=0}^{n-k-2} 2^{n-k-r-2}\binom{k+r}{r} \\ &= \sum_{r=0}^{n-k-1} 2^{n-k-r+1}\binom{k+r-3}{r} - \binom{n-1}{k} - \binom{n-2}{k-1} - 2\binom{n-3}{k-2} \\ &= \sum_{r=0}^{n-k-2} 2^{n-k-r+1}\binom{k+r-3}{r} - \binom{n-2}{k} - 2\binom{n-3}{k-1} - 4\binom{n-4}{k-2} . \qquad (n \geq k+2) \end{aligned} \tag{13}$$

Since

$$\sum_{n=k+2}^{\infty} x^{n-k-2} \sum_{r=0}^{n-k-2} 2^{n-k-r-2} \binom{k+r}{r}$$

$$= \sum_{r=0}^{\infty} \binom{k+r}{r} x^r \sum_{n=0}^{\infty} 2^n x^n = \frac{1}{(1-x)^{k+1}(1-2x)}$$

and

$$\sum_{n=k+2}^{\infty} x^{n-k-2} \left\{ \sum_{r=0}^{n-k-2} 2^{n-k-r+1} \binom{k+r-3}{r} - \binom{n-2}{k} - 2\binom{n-3}{k-1} - 4\binom{n-4}{k-2} \right\}$$

$$= \frac{8}{(1-x)^{k-2}(1-2x)} - \frac{1}{(1-x)^{k+1}} - \frac{2}{(1-x)^k} - \frac{4}{(1-x)^{k-1}}$$

$$= \frac{1}{(1-x)^{k+1}(1-2x)} .$$

it is evident that (13) holds

3. Put

$$\psi_n(y) = \sum_{k=0}^{n} \phi_k(n)\, y^k ,$$

so that

$$\psi_0(y) = 1, \qquad \psi_1(y) = 1 + y, \qquad \psi_2(y) = 2 + 2y + 2y^2$$

Then by (1) and (3), for $n \geq 2$,

$$\psi_n(y) = p(n) + \sum_{k=1}^{n-1} \{\phi_k(n-1) + \phi_{k-1}(n-1)\} y^k + 2^{n-1} y^n$$

$$= p(n) + (\psi_{n-1}(y) - p(n-1)) + y(\psi_{n-1}(y) - 2^{n-2} y^{n-1}) + 2^{n-1} y^n .$$

Thus

$$(14)\quad \psi_n(y) = p(n) - p(n-1) + (1+y)\psi_{n-1}(y) + 2^{n-2}y^n \qquad (n \geq 2).$$

For example,

$$\psi_2(y) = 1 + (1+y)^2 + y^2 = 2 + 2y + 2y^2$$

$$\psi_3(y) = 1 + (1+y)(2 + 2y + 2y^2) + 2y^3$$

$$= 3 + 4y + 4y^2 + 4y^3 \quad .$$

It is also evident from (14) that

$$(15)\quad \psi_n(1) = p(n) - p(n-1) + 2^{n-2} + 2\psi_{n-1}(y) \qquad (n \geq 2)$$

and

$$(16)\quad \psi_n(-1) = p(n) - p(n-1) + (-1)^n 2^{n-2} \qquad (n \geq 2)\ .$$

The last two formulas are also implied by (7).

REFERENCE

1. C. C. Cadogan, "On Partly Ordered Partitions of a Positive Integer," Fibonacci Quarterly, Vol. 9, 1971, pp. 329-336.