LINEAR HOMOGENEOUS DIFFERENCE EQUATIONS

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1. LINEAR HOMOGENEOUS DIFFERENCE EQUATIONS

Since its founding, this quarterly has essentially devoted its effort towards the study of recursive relations described by certain difference equations. The solutions of many of these difference equations can be expressed in closed form, not seldom referred to as Binet forms.

A previous article [2, p. 41] offered a closed form solution for the linear homogeneous difference equation

$$\sum_{j=0}^{N} A_j y(t + j) = 0 , \tag{1.1}$$

where

$$y(t) = a_n ; \quad n \le t < n + 1 ; \quad n = 0, 1, 2, \cdots$$

with the characteristic equation

$$\sum_{j=0}^{N} A_j z^j = 0 \tag{1.2}$$

expressed as

$$\prod_{j=0}^{N} (z - r_j) = 0 \tag{1.3}$$

with distinct roots r_j. The method of solution involved the use of Laplace Transforms. It was noted after the appearance of that article that many linear homogeneous difference equations actually encountered in practice do not have distinct roots to the characteristic equation (1.2). In other words, Eq. (1.2) is often of the form

$$(1.4) \qquad \prod_{i=1}^{M} (z - r_i)^{m_i} = 0 \qquad \left(N = \sum_{i=1}^{M} m_i \right) ,$$

where m_i is the multiplicity of the root r_i.

With respect to Laplace Transforms, the problem of handling multiple roots lies in the inversion of the transform $Y(s)$. It has been suggested that the definition of a "Maclaurin Series" could be regarded as a transform pair

$$(1.5) \qquad \begin{aligned} G(w) &= \sum_{t=0}^{\infty} [y(t)] \frac{w^t}{t!} \\ y(t) &= D_w^t [G(w)] \Big|_{w=0} \end{aligned}$$

which has the property that the transform of $y(t + j)$ is $G^{(j)}(w)$. Since the solution of linear homogeneous differential equations is already well known when involving multiple roots [1, p. 46], it was a straightforward procedure to establish the form for the complementary problem for difference equations.

The Laplace Transform of Eq. (1.1) given in [2, p. 44] is

$$(1.6) \qquad Y(s) = \left\{ \frac{e^s - 1}{s} \right\} \frac{\sum_{j=1}^{N} A_j \sum_{k=0}^{j-1} a_k e^{s(j-k-1)}}{\sum_{j=0}^{N} A_j e^{sj}} ,$$

and can be broken up into parts using the following theorem.

Theorem 1. (The Heaviside Theorem) If

$$Q(z) = \sum_{i=1}^{M} (z - r_i)^{m_i} ,$$

then

$$\frac{P(z)}{Q(z)} = \sum_{i=1}^{M} \sum_{j=1}^{m_i} \frac{C_{ij}}{(z - r_i)^j} ,$$

where

$$C_{ij} = \lim_{z \to r_i} \frac{1}{(m_i - j)!} D_z^{m_i - j} \left\{ \frac{P(z)}{Q(z)} (z - r_i)^j \right\} .$$

The reader can verify the formula for C_{ij} by creating the expression being operated on, and carry out the differentiation and limit. The essence of this theorem, however, is that the transform Y(s) can be expressed in the form

$$\text{(1.7)} \qquad Y(s) = \frac{e^s - 1}{s} \sum_{i=1}^{M} \sum_{j=1}^{m_i} \frac{C_{ij}}{(c^s - r_i)^j} .$$

The inverse of each of these terms is given by the next theorem.

Theorem 2.

$$L\left\{ \binom{n}{j - 1} r^{n-j+1} \right\} = \frac{e^s - 1}{s} \frac{1}{(e^s - r)^j} ,$$

where

$$\binom{n}{j - 1} = 0 \quad \text{when} \quad n < j - 1$$

(r represents an arbitrary root) .

Proof. Since

$$L\left\{\binom{n}{j-1}r^{n-j+1}\right\} = \int_0^\infty f(t)c^{-st}dt$$

$$= \sum_{n=0}^{\infty}\int_n^{n+1}\binom{n}{j-1}r^{n-j+1}e^{-st}dt$$

$$= \left(\frac{1-e^{-s}}{s}\right)\sum_{n=j-1}^{\infty}\binom{n}{j-1}r^{n-j+1}c^{-sn},$$

we need only show that

$$\sum_{n=j-1}^{\infty}\binom{n}{j-1}r^{n-j+1}e^{-sn} = \frac{e^s}{(e^s-r)^j},$$

by induction. The formula is true for $j = 1$ since

$$\sum_{n=0}^{\infty}(re^{-s})^n = \frac{1}{1-re^{-s}} = \frac{c^s}{(e^s-r)}.$$

Assume now that it holds for $j = k$, that is

$$\sum_{n=k-1}^{\infty}\binom{n}{k-1}r^{n-k+1}e^{-sn} = \frac{e^s}{(e^s-r)^k}.$$

Differentiating once, term-by-term, with respect to r yields

$$\sum_{n=k-1}^{\infty}\binom{n}{k-1}(n-k+1)r^{n-k}e^{-sn} = \frac{ke^s}{(e^s-r)^{k+1}}$$

or

$$\sum_{n=k}^{\infty} \binom{n}{k} r^{n-k} e^{-sn} = \frac{e^s}{(e^s - r)^{k+1}} ,$$

and thus implies the truth of the formula for the $(k+1)^{st}$ case. As a result of this theorem, the more general solution for the linear homogeneous difference equation (1.1) is given by

$$(1.8) \qquad y(t) = \sum_{i=1}^{M} \sum_{j=1}^{m_i} C_{ij} \binom{n}{j-1} r_i^{n-j+1} ,$$

where the C_{ij} are given (by Theorem 1) as

$$C_{ij} = \lim_{z \to m_i} \frac{1}{(m_i - j)!} D_z^{m_i - j} \left\{ \frac{\sum_{k=1}^{N} A_k \sum_{\ell=0}^{k-1} a_\ell z^{k-\ell-1}}{\prod_{k=1}^{M} (z - r_k)^{m_k}} (z - r_i)^j \right\}$$

or, by re-ordering the double summation according to z,

$$(1.9) \quad C_{ij} = \lim_{z \to m_i} \frac{1}{(m_i - j)!} D_z^{m_i - j} \left\{ \frac{\sum_{k=0}^{N-1} \sum_{\ell=k+1}^{N} A_\ell a_{\ell-k-1} z^k}{\prod_{k=1}^{M} (z - r_k)^{m_k}} (a - r_i)^j \right\} .$$

2. CONVOLUTION OF FIBONACCI SEQUENCES

The following problem related to the previous discussion was brought to my attention by Prof. V. E. Hoggatt, Jr. Initially, we are given that a convolution of a Fibonacci sequence is described by

$$H_{n+2} - H_{n+1} - H_n = F_n , \tag{2.1}$$

where F_n is the famous n^{th} Fibonacci number. The problem is to find a closed form (Binet form) for H_n. Since F_n satisfies the relationship

$$F_{n+2} - F_{n+1} - F_n = 0 , \tag{2.2}$$

Eq. (2.1) can be made homogeneous by substitution; that is, Eq. (2.2) can be re-written as

$$(H_{n+4} - H_{n+3} - H_{n+2}) - (H_{n+3} - H_{n+2} - H_{n+1}) - (H_{n+2} - H_{n+1} - H_n) = 0$$

or, collecting terms,

$$H_{n+4} - 2H_{n+3} - H_{n+2} + 2H_{n+1} + H_n = 0 . \tag{2.3}$$

Since $F_0 = 0$ and $F_1 = 1$, the starting values depending on H_0 and H_1 are

$$\begin{aligned} H_0 &= H_0 \\ H_1 &= H_1 \\ H_2 &= H_0 + H_1 \\ H_3 &= 1 + H_0 + 2H_1 . \end{aligned}$$

The characteristic equation of the difference relation (2.3) is

$$z^4 - 2z^3 - z^2 + 2z + 1 = 0$$

or

$$(z - \alpha)^2(z - \beta)^2 = 0 ,$$

where α is the well known golden ratio and β is the conjugate,

$$\alpha = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1 - \sqrt{5}}{2} .$$

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