SUBSEMIGROUPS OF THE ADDITIVE POSITIVE INTEGERS

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1. INTRODUCTION

Many of the attempts to obtain representations for commutative and/or Archimedean semigroups involve using the additive positive integers or subsemigroups of the additive positive integers. In this regard note references [1], [3], and [4]. The purpose of this paper is to catalogue the results that are known and to present some new results concerning the homomorphic images of such semigroups.

2. PRELIMINARIES

Let I denote the semigroups of additive positive integers. Lower case Roman letters will always denote elements of I. Subsemigroups of I will be denoted by capital Roman letters between J and Q inclusive. Results followed by a bracketed number and page numbers refer to that entry in the references and may be found there. Results not so identified are original and unpublished.

Theorem 1. ([2] pp. 36-48) Let K be a subsemigroup of I, then

i. There is $k \in I$ such that for $n \geq k$, $n \in K$ or

ii. There is $n \in I$, $n > 1$ such that n is a factor of all $k \in K$.

Proof. Suppose there exist $k_1, \cdots, k_m \in K$ such that the collection $(k_1, \cdots, k_m)$ has a greatest common divisor 1. Let K' be the subsemigroup of I generated by $\{k_1, k_2, \cdots, k_m\}$ clearly, $K' \subseteq K$. Let $k = 2k_1 \cdot k_2 \cdot \cdots \cdot k_m$ and for $b > k$, since the g.c.d. of $(k_1, \cdots, k_m)$ is one we may find integers $\alpha_1, \cdots, \alpha_m$ such that $\alpha_1 k_1 + \cdots + \alpha_m k_m = b$. (Note: the α_i are not necessarily positive.) We may now find integers q_i and r_i such that

$$\alpha_i = q_i k_1 \cdots k_{i-1} k_{i+1} \cdots k_m + r_i ,$$

where $0 < r_i \leq k_1 \cdots k_{i-1} \cdots k_m$ $(i = 2, 3, \cdots, m)$. Now let

$c_1 = \alpha_1 + (q_2 + \cdots + q_m)k_2k_3 \cdots k_m, \quad c_i = r_i, \quad (i = 2, 3, \cdots, m).$

We now have

$$b = c_1k_1 + c_2k_2 + \cdots + c_mk_m .$$

We have chosen $c_i \geq 0$ for $i = 2, 3, \cdots, m$. But since

$$c_2k_2 + \cdots + c_mk_m = r_2k_2 + \cdots + r_mk_m \leq k_1k_2 \cdots k_m \leq b,$$

clearly $c_1 \geq 0$. Thus every $b \geq k$ may be expressed as a linear combination of $\{k_1, \cdots, k_m\}$ where only positive integral coefficients are used.

If every finite sub collection of elements of K have g.c.d. greater than one, then clearly all of K have g.c.d. greater than one.]

Corollary 1. ([2] p. 39). Every K is finitely generated.

It is clear that there are essentially two types of subsemigroups of I:

i. Those that contain all integers greater than some fixed positive integer will be called relatively prime subsemigroups of I.

ii. Any other is a fixed integral multiple of a relatively prime subsemigroup.

Theorem 2. Let K, J be subsemigroups of I. Let the mapping $\mathbf{K}$ be a homomorphism from K onto J. Then $\mathbf{K}$ is in fact an isomorphism of K onto J of the type; for $k \in K$. $(k)\mathbf{K} = \gamma k$, where γ is a fixed rational number depending on K and J.

Proof. Since, by Corollary 1, K and J are finitely generated, let $(k_1, \cdots, k_m)$ be a generating set of K. Let $(j_1, \cdots, j_m)$ be the images in J of $(k_1, \cdots, k_m)$ under $\mathbf{K}$. Clearly $(j_1, \cdots, j_m)$. Now generate J.

$$(k_ik_1)\mathbf{K} = k_i(k_1)\mathbf{K} = k_ij_1$$

since $\mathbf{K}$ preserves positive integral multiples, but we also have

$$(k_ik_1)\mathbf{K} = (k_i)\mathbf{K}k_1 = j_ik_1$$

and

$$k_i j_1 = j_i k_1$$

so that

$$j_1 / k_1 k_i .]$$

Clearly for a given subsemigroup K not any rational number γ will do. Note that:

$$j_i = \frac{j_1}{k_1} k_i ,$$

but j_i is an integer and, k_1 divides k_i. If the collection $(k_1, \cdots, k_m)$ have greatest common divisor equal to one, then clearly γ is an integer. If the collection $(k_1, \cdots, k_m)$ have greatest common divisor $n \neq 1$, then $(k_1/n, \cdots, k_m/n)$ generates a relatively prime subsemigroup of I, call it K', and K and J are such that

$$K = nK', \qquad L = \gamma n K' ,$$

where γn is an integer. We have now shown:

Corollary 2. Let K and J be subsemigroups of I. For J any homomorphic image of K, K and J are integral multiples of a relatively prime subsemigroup, K', of I.

3. HOMOMORPHISMS

The results of Section 2 make it clear that no subsemigroup of I has a proper homomorphic image contained in I. Let us now examine the proper homomorphic images of subsemigroups of I.

Lemma 1. Let K be a relatively prime subsemigroup of I. Let $\sim$ be a congruence defined on K and satisfying:

$$\ni x, \quad y \in K, \quad x \neq y \text{ and } x \sim y .$$

Then, $K/\sim$ is finite.

Proof. Since K is relatively prime there is a least $k \in K$ such that for all $n \geq k$, $n \in K$. Suppose $x < y$ and at $y - x = m$. Now,

$$x + k \sim x + k + im, \qquad i = 1, 2, 3, \cdots$$

since by induction

$$x + k \sim (x + m + k = y + k)$$

and if $x + k \sim x + k + im$, then

$$x + k \sim x + h + (i + 1)m$$

by using the strong form of induction and adding $k + (i)m$ to both sides of: $x \sim x + m$. Clearly then, $x + m + h + 1$ is an upper bound for the order of $K/\sim$.]

Lemma 2. For K, k as in Lemma 1, let n be the least positive integer such that: for $x, y \in K$, $x \sim y$ and $x - y = n$. Then, for any c, $d \in K$, if $c \sim d$, $c < d$, $d - c = m$: we have $d - c = jn$.

Proof. (Let a be the least element of K such that $a \sim a + n$). We may find $k' \in K$ such that $c + k' > a + k$. Thus by Lemma 1, $c + k'$ is in one of the classes determined by

$$a + k, \ a + k + 1, \ \cdots, \ a + k + n - 1.$$

Thus

$$c + k' = a + k + jn + i,$$

and

$$c + k' + m = a + k + j'n + i',$$

but $c + k' + m \sim c + k'$, and $a + k + j'n + i \sim a + k + jn + i'$, but this gives $a + k + i \sim a + k + i'$. Thus, $i = i'$ since n is the least positive integral difference of equivalent elements of K.]

For finite homomorphic images of subsemigroups of I, call n, as defined in Lemma 2, the period of the congruence.

Lemma 3. Let K, k, n, a be as in Lemma 2. Let be a congruence on K such that for $c \sim d$, $d > c$, $d - c \in K$. Then $K/\sim$ has exactly n non-singleton classes.

Proof. Let $d - c = m$. Then by Lemma 2, $m = jn$. We have $jn \in K$ and for p sufficiently large $c + (p)jn > a + k$. Thus, $c + (p)jn \sim a + k + i$ for some i; $0 \leq i \leq n - 1$. But since $jn \in K$, $c + (p)jn \sim c$ for $p = 1, 2, 3, \cdots$. Thus $c \sim a + k + i$ and the non-singleton classes may be represented by $a + k$, $a + k + 1, \cdots, a + k + n - 1$.]

If c is an element of a relatively prime K, where $c \sim a + k + i$ (a, k being as in Lemma 2) then if $\sim$ has period n we have: $c \equiv a + ki \pmod{n}$. This follows immediately from Lemma 2.

Congruences on a relatively prime K which fail to satisfy the conditions of Lemma 3 may be described as follows. There are the n classes represented by $a + h$, $a + h + 1, \cdots, a + h + n - 1$; there are any number of singleton classes for elements between $a + h$ and the least element of K. There may be finite non-singleton classes of elements between $a + h$ and the least element of K, but from Lemma 3 no two elements in a finite class may differ by an element of K.

4. SUBSEMIGROUPS OF CYCLIC SEMIGROUPS

In this section we treat subsemigroups of finite cyclic semigroups. Let R be the finite cyclic semigroup of index r and period m. Elements of R will be represented by integers; R will be written additively.

Lemma 1. Let T be the subsemigroup of R generated by the elements $t_1, t_2, \cdots, t_k$. If the greatest common divisor of $\{t_1, t_2, \cdots, t_k, m\}$ is one, then T contains the periodic part of R.

Proof. Let t' be the g.c.d. of $\{t_1, t_2, \cdots, t_h\}$. By Theorem 1, Section 2, the subsemigroup of I generated by $\{t_1/t', t_2/t', \cdots, tk/t'\}$ contains all integers greater than some fixed integer k. But for some p all $q \geq p$ are such that $qt' > k$. Now let

$$(k + i)t' - r \equiv_m (k + j)t' - r ,$$

then $(nj - in)t' = n'm$, but t' and m are relatively prime. Thus, m divides $nj - in$.]

The remainder of the subsemigroup of R generated by $\{t_1, t_2, \cdots, t_h\}$ is the intersection in R of the subsemigroup of I generated by the t_i considered as integers. If the g.c.d. of $\{t_1, t_2, \cdots, t_k, m\} = p > 1$, then the subsemigroup generated contains m/p elements of the periodic part of R, and can thus be made isomorphic to a subsemigroup of the type described in Lemma 1 by changing the period of R to m/p.

Finally, let K be the subsemigroup of I generated by $\{t_1, t_2, \cdots, t_k\}$ considered as integers, where $t_1, t_2, \cdots, t_h \in R$ a finite cyclic semigroup of index r and period m, and the g.c.d. of $\{t_1, t_2, \cdots, t_h, m\}$ is one. Let $K' = K \cup N$, where N is all of I greater than r. Clearly K' is a subsemigroup of I. Let $\sim_r$ be the relation:

$$x,\ y \in K',\ x \sim_r y = x = y \quad \text{or} \quad (x, y \geq r \ \text{and} \ x \equiv_m y)\,.$$

The relation $\sim_r$ is a congruence on K'. Now identify the elements of $K'/\sim_r$ with the elements of the subsemigroup of R generated by $\{t_1, \cdots, t_h\}$ in the natural way. We then have:

Theorem 2. The semigroup $K'/\sim_r$ is isomorphic to the subsemigroup of R generated by $\{t_1, t_2, \cdots, t_h\}$.

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