A GENERAL Q-MATRIX

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1. INTRODUCTION

Let F_n be the n^{th} Fibonacci number and let

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

This matrix has the interesting property that

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}.$$

In this paper, we introduce a general type of Q-matrix for the generalized Fibonacci sequence $\{f_{n,r}\}$, and some of the interesting properties of the Q-matrix are then generalized for these sequences. An extension to the general linear recurrent sequence is also given. See [1] for more information on the Q-matrix proper.

2. THE MATRIX Q_r

Recall that the Fibonacci numbers $\{F_n\}$ are defined by $F_{n+2} = F_{n+1} + F_n$, with $F_0 = 0$, $F_1 = 1$. Now let us define the generalized Fibonacci sequences $\{f_{n,r}\}$ for $r \ge 2$ by $f_{n,r} = f_{n-1,r} + \cdots + f_{n-r,r}$, with $f_{0,r} = f_{1,r} = \cdots = f_{r-2,r} = 0$, $f_{r-1,r} = 1$. Note that $r = 2$ gives the Fibonacci numbers.

Now define a matrix Q_r by

$$Q_r = \begin{pmatrix} 1 & 1 & 0 & & \cdots & 0 \\ 1 & 0 & 1 & & \cdots & 0 \\ \vdots & 0 & 0 & 1 & \cdots & 0 \\ \cdot & 0 & & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Note that Q_r is just the $r-1$ identity matrix bordered by the first column of 1's and last row of 0's. In order to motivate this definition, note that

$$\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} Q_2 = \begin{pmatrix} F_{n+2} & F_{n+1} \\ F_{n+1} & F_n \end{pmatrix} .$$

We have thus defined Q_r so that this property holds for the matrix

$$(f_{n+r+1-i-j,r}), \qquad 1 \le i,\ j \le r .$$

Theorem 1.

$$Q_r^n = \begin{pmatrix} f_{n+r-1,r} & f_{n+r-2,r} & \cdots & f_{n,r} \\ \sum_{i=0}^{r-2} f_{n+r-2-i,r} & \sum_{i=0}^{r-2} f_{n+r-3-i,r} & \cdots & \sum_{i=0}^{r-2} f_{n-i-1,r} \\ \vdots & \vdots & & \vdots \\ f_{n+r-2,r} & f_{n+r-3,r} & \cdots & f_{n-1,r} \end{pmatrix}$$

(the general term is

$$q_{jk} = \sum_{i=0}^{r-j} f_{n+r-i-k-1,r} \qquad).$$

Proof. Let r be fixed and use induction on n. This is trivially verified for $n = 1, 2$. Assume true for n, and consider

$$Q_r^{n+1} = Q_r^n Q_r = \begin{pmatrix} f_{n+r-1,r} & f_{n+r-2,r} & \cdots & f_{n,r} \\ \sum_{i=0}^{r-2} f_{n+r-2-i,r} & \sum_{i=0}^{r-2} f_{n+r-3-i,r} & \cdots & \sum_{i=0}^{r-2} f_{n-i-1,r} \\ \vdots & \vdots & & \vdots \\ f_{n+r-2,r} & f_{n+r-3,r} & \cdots & f_{n-1,r} \end{pmatrix} .$$

(equation continued on next page.)

$$\cdot\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ . & 0 & 0 & 1 \cdots & 0 \\ \vdots & 0 & & \cdots & 0\ 1 \\ 1 & 0 & 0 & 0 \cdots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} f_{n+r,r} & f_{n+r-1,r} & & f_{n+1,r} \\ \sum_{i=0}^{r-2} f_{n+r-i-1,r} & \sum_{i=0}^{r-2} f_{n+r-2-i,r} & \cdots & \sum_{i=0}^{r-2} f_{n-i,r} \\ \vdots & \vdots & & \vdots \\ f_{n+r-1,r} & f_{n+r-2,r} & & f_{n,r} \end{pmatrix} = Q_r^{n+1},$$

which completes the proof of the theorem.

We write this matrix in neater form by letting $P_r = (f_{r-i-j+2,r})$, $1 \le i,\ j \le r$, where $f_{-n,r}$ is found by the recursion relationship. Then

$$P_r Q_r^n = \begin{pmatrix} f_{n+r,r} & \cdots & f_{n+1,r} \\ \vdots & & \vdots \\ f_{n+1,r} & \cdots & f_{n-r+2,r} \end{pmatrix}.$$

An interesting special case of our theorem occurs when $r = 3$, where the numbers $\{f_{n,3}\}$ are the so-called Tribonacci numbers of Mark Feinberg.

3. APPLICATIONS

We now develop some of the interesting properties of the matrices Q_r^n and $P_r Q_r^n$, which in turn are generalizations of interesting properties of the matrix Q^n, which is the special case when $r = 2$.

It is readily calculated that

$$\det (P_r Q_r^n) = (\det P_r)(\det Q_r)^n = (-1)^{(2n+r)(r-1)/2}.$$

For $r = 2$, we have the corresponding Fibonacci identity

$$F_{n+1} F_{n-1} - F_n^2 = (-1)^{n+1}.$$

The traces of Q_r^n and $P_r Q_r^n$ are also readily seen to be

$$\mathrm{Tr}(Q_r^n) = \sum_{j=1}^{r}\left(\sum_{i=0}^{r-j} f_{n+r-i-j-1,r}\right) = \sum_{j=1}^{r} j f_{n+r-j-1,r}$$

$$\mathrm{Tr}(P_r Q_r^n) = f_{n+r,r}^{-} + f_{n+r-2,r} + \cdots + f_{n-r+2,r}\,.$$

For $r = 2$, we have

$$\mathrm{Tr}(Q^n) = F_{n-1} + F_{n+1} = L_n\,.$$

The characteristic polynomial of Q_r is $x^r - x^{r-1} - \cdots - x - 1$, which is also the auxiliary polynomial for the sequence $\{f_{n,r}\}$. Since Q_r satisfies its own characteristic equation, $Q_r^r = Q_r^{r-1} + \cdots + Q_r + I$, hence

$$Q_r^{rn} = (Q_r^{r-1} + \cdots + Q_r + I)^n\,.$$

Expanding by the multinomial theorem and equating elements in the upper right-hand corner yields

$$f_{rn,r} = \sum_{\substack{k_1,\cdots,k_r \\ k_1+\cdots+k_r=r}} \frac{n!}{k_1!\cdots k_r!} f_{k_1+2k_2+\cdots+(r-1)k_{r-1},r}$$

For $r = 2$, we recover the familiar

$$F_{2n} = \sum_{k=0}^{n}\binom{n}{k}F_k\,.$$

Now consider the matrix equation $Q_r^{m+n} = Q_r^m Q_r^n$; equating elements in the upper left-hand corner yields

$$f_{m+n+r-1,r} = \sum_{j=1}^{r}\left(\sum_{i=0}^{r-j} f_{m+r-j,r}\, f_{n+r-2-i,r}\right)$$

and for $r = 2$, we have $F_{m+n+1} = F_{m+1}F_{n+1} + F_mF_n$. Note that several other general identities can be obtained in this way.

We now use the matrix Q_r to show that the product of two elements of finite order in a non-abelian group is not necessarily of finite order. This generalizes a counterexample given by Douglas Lind in [2], which results for $r = 2$. Let

$$R_r = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}, \qquad S_r = \begin{pmatrix} -1 & -1 & & & 0 \\ 1 & & 1 & & \\ \vdots & & 0 & \ddots & \\ \vdots & & & & 1 \\ 1 & & & & 0 \end{pmatrix}$$

be elements of the group of invertible square matrices, then

$$R_r^2 = S_r^{r+1} = I \;,$$

so R_r and S_r are of finite order, but $(R_rS_r)^n = Q_r^n \neq I$, for all n, by Theorem 1, so that R_rS_r is not of finite order.

It is of some interest to observe that the matrices Q_r^n give explicit examples of Anosov toral diffeomorphisms. That is, viewed as a linear map on $\mathbb{R}^r$, Q_r^n preserves integer points and is invertible with $\det \neq 1$, hence induces a diffeomorphism on the quotient space $\mathbb{R}^r/\mathbb{Z}^r$. The hyperbolic toral structure follows, since Q_r^n has no eigenvalue of modulus 1, using an argument via the characteristic polynomial as in [3]. Any such Anosov toral diffeomorphism comes from a linear recurrent sequence whose auxiliary equation is given by the polynomial of the diffeomorphism.

4. THE GENERAL LINEAR RECURRENT SEQUENCE

We now show how a Q-type matrix can be determined for the general r^{th} order linear recurrence relation

$$u_{n+r,r} = a_{r-1}u_{n+r-1,r} + \cdots + a_0u_{n,r}$$

with initial values $u_i = b_i$, $i = 0, 1, \cdots, r-1$, where $b_0, b_1, \cdots, b_{r-1}$

are arbitrary constants. This is done in a sequence of successive generalizations.

Define a sequence $\{f^*_{n,r}\}$ by $f_{n+r,r} = f^*_{n+r-1} + \cdots + f^*_{n,r}$, with initial values $f^*_i = b_i$, $0 \le i \le r-1$. (Note that $b_0 = b_1 = \cdots = b_{r-2} = 0$, $b_{r-1} = 1$ give the $\{f_{n,r}\}$ defined previously.) To find a Q-type matrix for the $\{f^*_{n,r}\}$, we need the following identity:

$$f^*_{n,r} = \sum_{i=1}^{r} b_{i-1} \sum_{j=1}^{i} f_{n-j-1,r} ,$$

which is easily proved by induction on n. Now let $B = (b_{r-1} \cdots b_0)$, then

$$BQ_r^n = (f^*_{n+r,r} \cdots f^*_{n+1,r}) ,$$

$$BQ_r^{n-1} = (f^*_{n-r+1}, \cdots, f^*_{n,r}), \cdots, BQ_r^{n-r+1} = (f^*_{n+1,r} \cdots f^*_{n-r+2,r}) ,$$

by our identity. Thus, we have the following Q-type matrix for our sequence $\{f^*_{n,r}\}$:

$$(Q_r^*)^n = \begin{pmatrix} BQ_r^n \\ \vdots \\ BQ_r^{n-r+1} \end{pmatrix} = \begin{pmatrix} f^*_{n+r,r} & \cdots & f^*_{n+1,r} \\ \vdots & & \vdots \\ f^*_{n+1,r} & \cdots & f^*_{n-r+2,r} \end{pmatrix} .$$

Now consider the sequences $\{u^*_{n,r}\}$ defined by

$$u^*_{n+r,r} = a_{r-1}u^*_{n+r-1,r} + a_{r-2}u^*_{n+r-2,r} + \cdots + a_0 u^*_{n,r} ,$$

with initial values $u^*_{n,r} = 0$, $0 \le n \le r-2$, $u^*_{r-1,r} = 1$. As in Theorem 1, we have the following Q-type matrix for the sequence $\{u^*_{n,r}\}$:

$$(R_r^*)^n = \begin{pmatrix} u^*_{n+r-1,r} & u^*_{n+r-2,r} & \cdots & u^*_{n,r} \\ \sum_{i=0}^{r-2} a_{r-2-1}u^*_{n+r-2-1,r} & \sum_{i=0}^{r-2} a_{r-2-i}u^*_{n+r-3-i,r} & \cdots & \sum_{i=0}^{r-2} a_{r-2-i}u^*_{n-i-1,r} \\ \vdots & \vdots & & \vdots \\ a_0\, u^*_{n+r-2,r} & a_0\, u^*_{n+r-3,r} & \cdots & a_0\, u^*_{n-1,r} \end{pmatrix}$$

which is proved by induction on n.

We now piece these two partial results together to derive a general Q-type matrix for the general linear recurrent sequence $\{u_{n,r}\}$ defined in the beginning of this section. To do this, we need the following identity:

$$u_{n,r} = \sum_{i=1}^{r} b_{i-1} \sum_{j=1}^{i} a_{i-j}\, u^*_{n-j-1,r} \quad ,$$

which is proved by induction. As before, let $B = (b_{r-1} \cdots b_0)$, then by our identity,

$$B(R_r^*)^n = (u_{n+r,r} \cdots u_{n+1,r}), \cdots, B(R_r^*)^{n-r+1} = (u_{n+1,r} \cdots u_{n-r+2,r}) .$$

Hence, we have the following.

Theorem 2.

$$(R_r)^n = \begin{pmatrix} B(R_r^*)^n \\ \vdots \\ B(R_r^*)^{n-r+1} \end{pmatrix} = \begin{pmatrix} u_{n+r,r} & \cdots & u_{n+1,r} \\ \vdots & & \\ u_{n+1,r} & \cdots & u_{n-r+2,r} \end{pmatrix}$$

Thus, there is a general Q-type matrix for any linear recurrent sequence.

REFERENCES

1. V. E. Hoggatt, "A Primer for the Fibonacci Numbers," Fibonacci Quarterly, Vol. 1, No. 3, Oct., 1963, pp. 61-65.

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