MAKING GOLDEN CUTS WITH A SHOEMAKER'S KNIFE

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1. The problem of finding the "golden cut" (or section) of a line segment was known to the early Greeks and is solved in Euclid II, 11 [1]: let segment AB be divided into two segments by the point G such that AB/AG : AG/GB. Then G is the golden cut of AB, and AG/GB, the golden ratio. The "shoemaker's knife" (or arbelos) was the name given by Archimedes [2] to the following figure: let K be any point on segment CD and let semicircles be drawn on the same side of CD, with CD, CK, and KD as diameters. The figure whose boundary consists of these semicircles is called a shoemaker's knife (see Fig. 1).

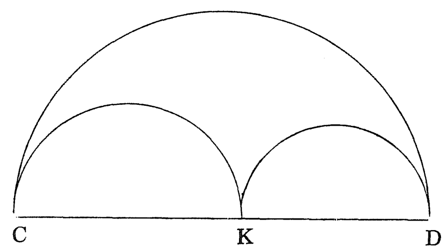

Figure 1

In this note, we will show how, given a golden cut G in a segment AB, we can, with the aid of an arbelos, generate a golden cut on any segment with length smaller than AB, in a swift and straightforward fashion. This in itself should justify bestowing the title of "golden" on the arbelos; however, we will also give a justification which conforms more with historical criteria.

2. Let segment AB be given and let G denote the golden cut of AB. Let C be any interior point of segment AB. The problem is to find the golden cut of segment AC. Construct a semicircle with diameter AB. Locate point D on the semicircle such that the distance AD is equal to AC. Draw chord AD, and drop a perpendicular from G to AD. Denote the foot of the perpendicular by G'. (See Fig. 2.)

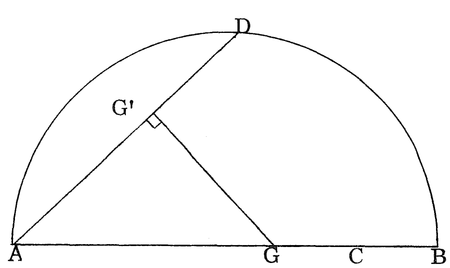

Figure 2

Claim. G' is the golden cut of chord AD.

Proof. Draw chord BD. Since angle ABD is a right angle, it follows that the triangles $AG'G$ and ADB are similar, from which it follows that $AG'/G'D = AG/GB$. Hence, G' is the golden cut of AD, which was to be shown. Finally, by locating point G'' on segment AB such that $AG'' = AG'$, we have that G'' is the golden cut of segment AC.

Since angle $AG'G$ is a right angle, it follows that the points A, G', G determine a semicircle. Thus, we have the following corollary: let AD be any chord in the semicircle with diameter AB. Let G' denote the golden cut of AD. Then the locus of all such points G' is a semicircle with diameter AG (G, the golden cut of AB).

It is easy to see that both the construction and above corollary carry over with obvious modifications if we reference everything at point B, rather than point A. Thus, if BD is any chord in the semicircle, and H its golden cut, then the locus of all such points H is a semicircle with GB as diameter.

The figure consisting of the semicircles with diameters AB, AG, and GB is, of course, an arbelos. Now we are ready to reverse the above procedure and deal with the main problem: viz, to utilize the shoemaker's knife to effect golden cuts. Let AB be a given line segment and G its golden cut. Draw semicircles with diameters AB, AG, and GB, respectively. Let D be any point on the semicircle with diameter AB, and draw chord AD intersecting the semicircle on diameter AG at G'. Then G' is the golden cut of AD. (See Fig. 3.)

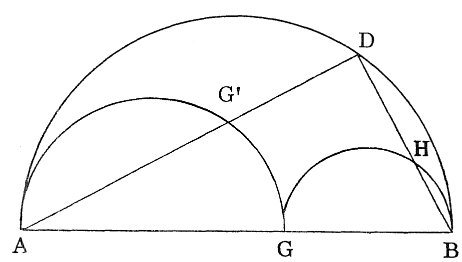

Figure 3

By drawing chord BD intersecting the semicircle on diameter GB at H, we see also that H is the golden cut of BD. The argument for both these statements is the same as that given above.

In light of the latter argument, it is reasonable that the arbelos in Fig. 3 should be termed "golden." To see that this terminology is in fact also historically justified, observe that the arbelos can be viewed as a continuous deformation of a right triangle, where the hypotenuse corresponds to the largest semicircle and the legs to the smaller semicircles. Historically [1], a right triangle is called golden if the ratio of its legs is the golden ratio. In light of the above observation, it would be in keeping to term the arbelos "golden" if the ratio of its "legs" (i.e., its minor semicircles) were the golden ratio. A simple computation for our arbelos reveals that this is in fact the case; the length of the semicircle on AG is evi-

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