A NOTE ON THE NUMBER OF FIBONACCI SEQUENCES

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In an article entitled "On the Ordering of Fibonacci Sequences" [1], the author pointed out that if we consider Fibonacci sequences with relatively prime successive terms and a series of positive terms extending to the right, there is (apart from the case of the Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, ⋯), one point in the sequence and only one where a positive term is less than half the next positive term. Such being the case, it is convenient to identify a Fibonacci sequence by these two numbers, as this gives a unique means of specifying a sequence.

The present note is concerned with this question: If the two identifying numbers of a Fibonacci sequence as presently defined are less than or equal to a positive integer m, how many Fibonacci sequences does this give?

Theorem. If the starting numbers of a Fibonacci sequence are $\leq m$ $(m \geq 2)$, the number of Fibonacci sequences that can be formed is:

$$1/2 \sum_{k=1}^{m} \phi(k)$$

where $\phi(m)$ is Euler's totient function.

Proof. The following table indicates the situation for small values of m and serves as the basis of the subsequent mathematical induction

m	$\phi(m)$	$\Sigma\phi(k)$	$\frac{1}{2}\Sigma\phi(k)$	Sequences
1	1			
2	1	2	1	(1,1)
3	2	4	2	(1,3)
4	2	6	3	(1,4)
5	4	10	5	(1,5), (2,5)
6	2	12	6	(1,6)
7	6	18	9	(1,7), (2,7), (3,7)

Within the limits of this table, it is clear that the total number of sequences that may be formed for any given m is $\frac{1}{2}\Sigma\phi(k)$.

Assume that this is true to some given m. If we enlarge the domain by including $m + 1$, the new sequences added will be those involving this quantity as well as those quantities less than half of $m + 1$ and relatively prime to it. But the number of such quantities is $\frac{1}{2}\phi(m+1)$. Thus it follows that if the formula is true for m, it is true for $m + 1$ and the theorem is proved in general.

REFERENCE

1. Brother U. Alfred, "On the Ordering of Fibonacci Sequences," Fibonacci Quarterly, Dec. 1963, pp. 43-46.

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That is, we have shown that

$$C_k(x) = A_k(x) \cdot (1 - x)^{-\frac{1}{2}k(k+1)-1} \qquad (4.8)$$

where $A_k(x)$ is a polynomial in x of degree $\frac{1}{2}k(k-1)$ given by either of

$$A_k(x) = \sum_{j=k}^{\frac{1}{2}k(k+1)} a_{kj}(1 - x)^{\frac{1}{2}k(k+1)-j} \qquad (4.9)$$

or

$$A_k(x) = \sum_{j=k}^{\frac{1}{2}k(k+1)} a_{kj}\, x^{j-k}(x - 1)^{\frac{1}{2}k(k+1)-j} . \qquad (4.10)$$

Notice that the symmetry property (1.9) follows by comparing (4.9) and (4.10). The first few values of $A_k(x)$ are $A_1(x) = 1$, $A_2(x) = 1 + x$, $A_3(x) = 1 + 7x + 7x^2 + x^3$.

REFERENCES

1. L. Carlitz and John Riordan, "Enumeration of Certain Two-Line Arrays," Duke Math. J., Vol. 32 (1965), pp. 529-539.
2. L. Carlitz and R. A. Scoville, Problem E2054, MAA Monthly, Vol. 75 (1968), p. 77.
3. P. A. MacMahon, Combinatory Analysis, Vol. 1, Cambridge, 1915.

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(3) Articles of standard size for which additional background material may be obtained.

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