A COUNTING FUNCTION OF INTEGRAL n-TUPLES

HWA S. HAHN
West Georgia College, Carrollton, Georgia

1. INTRODUCTION

Let P be the set of positive integers and let P^n be the set of n-tuples of positive integers. Many fresnmen books talk about how to count P^2 but rarely exhibit a counting function such as [2]

$$f_2(p_1, p_2) = p_1 + (p_1 + p_2 - 1)(p_1 + p_2 - 2)/2 .$$

E. A. Maier presented a counting function of P^n in this Quarterly [1]. In this note we show another more simple counting function of P^n and also discuss its inverse function and some applications.

2. THEOREM

The following polynomial in n variables

$$f_n(p_1, p_2, \cdots, p_n) = p_1 + \sum_{k=2}^{n} \binom{s_k - 1}{k} , \tag{1}$$

where

$$s_k = p_1 + p_2 + \cdots + p_k \quad \text{and} \quad \binom{s_k - 1}{k} = 0$$

for $s_k - 1 < k$, is a counting function of P^n.

Proof. Consider the set, call it the s-layer, of lattice points of positive coordinates $(x_1, x_2, \cdots, x_n)$ satisfying

$$x_1 + x_2 + \cdots x_n = s .$$

This s-layer contains

$$\binom{s-1}{n-1}$$

points. For, it is the number of ways of putting $n - 1$ markers in $s - 1$ spaces between 1's in

$$1 + 1 + \cdots + 1 = s .$$

Then the collection of s-layers, call it a pyramid, ranging $n \le s < s_n$, which is the largest pyramid without the given point $(p_1, p_2, \cdots, p_n)$, contains

$$\binom{n-1}{n-1} + \binom{n}{n-1} + \cdots + \binom{s_n - 2}{n-1}$$

points. But this sum is simply

$$\binom{s_n - 1}{n}.$$

For,

$$\binom{s_n - 1}{n} = \binom{s_n - 2}{n-1} + \binom{s_n - 2}{n}$$
$$= \binom{s_n - 2}{n-1} + \binom{s_n - 3}{n-1} + \binom{s_n - 3}{n}$$
$$= \cdots \qquad .$$

Next, we count points $(x_1, x_2, \cdots, x_n)$ such that

$$\sum x_i = s_n ,$$

up to $(p_1, p_2, \cdots, p_n)$. Since x_n is determined by $(x_1, x_2, \cdots, x_{n-1})$ and s_n, we need to count only $(n - 1)$-tuples from $(1, 1, \cdots, 1)$ to $(p_1, p_2, \cdots, p_{n-1})$. For this we may use the function $f_{n-1}(p_1, p_2, \cdots, p_{n-1})$.

Thus, we obtain

$$f_n(p_1, p_2, \cdots, p_n) = f_{n-1}(p_1, p_2, \cdots, p_{n-1}) + \binom{s_n - 1}{n} .$$

And this recursive formula gives

$$f_n(p_1, p_2, \cdots, p_n) = p_1 + \sum_{k=2}^{n} \binom{s_k - 1}{k} ,$$

(taking $f_1(p_1) = p_1$).

Notes. 1. For $s_0 = 1$,

$$f_n(p_1, p_2, \cdots, p_n) = \sum_{k=0}^{n} \binom{s_k - 1}{k} ,$$

which is a string of pyramids of each dimension from 0 to n.

2. From its counting method f_n is clearly 1 - 1. However, we can also prove as follows. If $(p_1, p_2, \cdots, p_n) \neq (p_1', p_2', \cdots, p_n')$, then there exists m such that $s_m \neq s_m'$ and $s_k = s_k'$ for $k > m$. Say, $s_m < s_m'$ (without loss of generality). Since $1 = s_0 \leq s_1 < \cdots < s_m \leq s_m' - 1$,

$$\sum_{k=0}^{m}\binom{s_k - 1}{k} \le \sum_{k=0}^{m}\binom{s_m - (m - k) - 1}{k} = \sum_{k=0}^{m}\binom{s_m - (m - k) - 1}{s_m - m - 1}$$

$$= \binom{s_m}{s_m - m} = \binom{s_m}{m} < p_1' + \binom{s_m' - 1}{m} \le \sum_{k=0}^{m}\binom{s_k' - 1}{k} .$$

These inequalities imply $f_n(p_1, \cdots, p_n) < f_n(p_1', \cdots, p_n')$. The following section also shows that f_n is onto.

3. THE INVERSE MAPPING $f_n^{-1} : P \longrightarrow P^n$

The following algorithm produces $s_n, s_{n-1}, \cdots, s_1 (=p_1)$ from a given positive integer p.

First, determine s_n satisfying

$$\binom{s_n - 1}{n} < p \le \binom{s_n}{n} .$$

Then $s_{n-1}, s_{n-2}, \cdots, s_1$ from

$$\binom{s_{n-1} - 1}{n - 1} < p - \binom{s_n - 1}{n} \le \binom{s_{n-1}}{n-1} ,$$

$$\binom{s_{n-2} - 1}{n - 2} < p - \binom{s_n - 1}{n} - \binom{s_{n-1} - 1}{n - 1} \le \binom{s_{n-2}}{n-2} ,$$

$$\cdots$$

$$\binom{s_1 - 1}{1} < p - \sum_{k=0}^{n}\binom{s_k - 1}{k} = \binom{s_1}{1} .$$

Thus

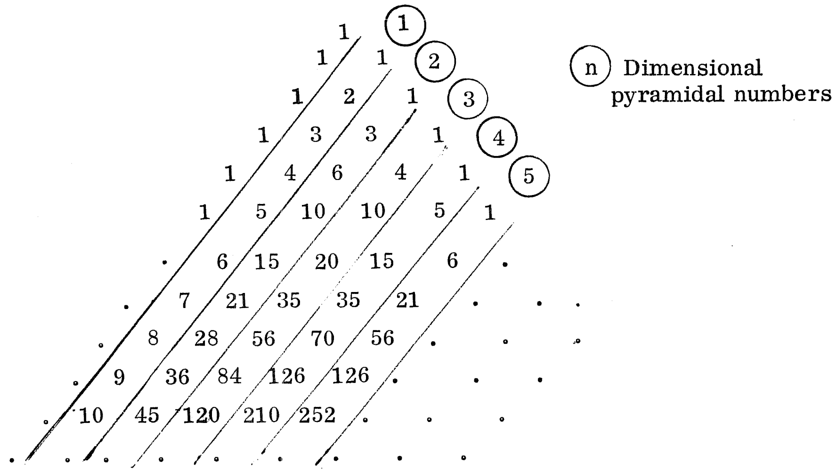
$$f_n - 1(p) = (s_1, s_2 - s_1, \cdots, s_n - s_{n-1}) ,$$

where

$$s_k - s_{k-1} = p_k \quad \text{for} \quad k > 1 \quad \text{and} \quad s_1 = p_1 .$$

4. PYRAMIDAL NUMBERS $\binom{s_n - 1}{n}$ IN PASCAL'S TRIANGLE

In the construction of the inverse image $f_n^{-1}(p)$ it is helpful to use Pascal's triangle, in which $(n + 1)^{st}$ diagonal line is the ordering of all n dimensional pyramids.



For example, to compute $f_3^{-1}(100)$ express 100 as a sum of pyramidal numbers of dimensions 3, 2, and 1 as follows:

$$100 = 84 + 15 + 1 = \binom{10-1}{3} + \binom{7-1}{2} + 1\,.$$

Then $s_3 = 10$, $s_2 = 7$, $s_1 = 1$ and thus

$$f_3^{-1}(100) = (1,\ 7-1,\ 10-7) = (1,\ 6,\ 3)\,.$$

5. COUNTING LATTICE POINTS IN EUCLIDEAN n-SPACE

Take any counting function of Z, the set of integers, for example f_0 defined by

$$f_0(z) = 2\delta z + \frac{1-\delta}{2}\,,$$

where

$$\delta = \begin{cases} 1 & \text{for } z > 0\,, \\ -1 & \text{for } z \le 0\,. \end{cases}$$

Then the ordinal number for $(z_1, z_2, \cdots, z_n)$ is given

$$f_n\bigl(f_0(z_1),\ f_0(z_2),\ \cdots,\ f_0(z_n)\bigr) = \sum_{k=0}^{n} \binom{S_k - 1}{k}\,,$$

where

$$S_k = \sum_{i=1}^{k} f_0(z_i)\,.$$

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