MORE ABOUT MAGIC SQUARES CONSISTING OF DIFFERENT PRIMES

EDGAR KARST
University of Arizona, Tucson, Arizona

Let a magic square of order n be surrounded by numbers such that square plus numbers form another magic square of order $n + 2$ and similar magic squares of order $n + 4$, $n + 6$, and so on; then the center square may be called a nucleus and the surrounding numbers a frame.

In a letter of August 8, 1971, V. A. Golubev concocts and gives permission to publish the following magic square of order 11 consisting of primes of the form $30x + 17$ and including similar magic squares of order 3, 5, 7, and 9.

GOLUBEV'S PRIME MAGIC SQUARE

73547	52757	52457	74567	51287	75767	49787	49727	24527	119087	72977
80177	59447	54767	71987	54167	72647	53597	50147	84407	68687	46457
80897	73127	67217	60527	60257	58427	59387	70937	66467	53507	45737
81077	53117	75437	64877	60497	54347	71147	65717	51197	73517	45557
81647	52727	55967	60017	64577	61637	63737	66617	70667	73907	44987
44927	74507	69737	72707	62477	63317	64157	53927	56897	52127	81707
44417	51257	57737	58067	62897	64997	62057	68567	68897	75377	82217
43787	101537	56957	60917	66137	72287	55487	61757	69677	25097	82847
84437	46187	60167	66107	66377	68207	67247	55697	59417	80447	42197
27917	57947	71867	54647	72467	53987	73037	76487	42227	67187	98717
53657	73877	74177	52067	75347	50867	76847	76907	102107	7547	53087

The nucleus of order 3 contains the elements 61637, 62057, $\cdots$, 64997 which are the nine primes in A. P. given in the appendix of [3]. A pair of opposite primes in each frame adds up to $126634 = 2 \cdot 63317$. Important for constructing the frames is the fact that the sums of two opposite sides without the corners must be the same. Hence, the frame of order 5 has

$$60497 + 54347 + 71147 + 66137 + 72287 + 55487 = 66617 + 53927 + 68567 + 60017 + \\ + 72707 + 58067 = 379902 = 2 \cdot 3 \cdot 63317 ,$$

the frame of order 7 has

$$60527 + 60257 + 58427 + 59387 + 70937 + 66107 + 66377 + 68207 + 67247 + 55697 \\ = 51197 + 70667 + 56897 + 68897 + 69677 + 75437 + 55967 + 69737 + 57737 + 56957 \\ = 633170 = 2 \cdot 5 \cdot 63317 ,$$

and so on. This comprises the simpler part of the construction. For the corners, the two pairs of diagonally opposite primes must each not only add up to $126634 = 2 \cdot 63317$, but the sum of the elements of each of the two diagonals must also agree with the magic constant already obtained by summing up n members in a vertical or horizontal way. This is the more difficult part of the construction. Is someone able to attach a frame of order 13 to Golubev's beautiful magic square of primes $30x + 17$?

If we have prime magic squares of odd order, it is not necessary that the nucleus consists of primes in A.P. such that

$$p_1 + d = p_2, \quad p_2 + d = p_3, \quad \cdots, \quad p_8 + d = p_9 .$$

In fact, the 3rd and 6th d in those equations may be replaced by any number $y = 6m$ such that the elements still remain primes. For example,

$$-17 + 6 = -11, \quad -11 + 6 = -5, \quad -5 + 12 = 7, \quad 7 + 6 = 13, \quad 13 + 6 = 19,$$

$$19 + 12 = 31, \quad 31 + 6 = 37, \quad 37 + 6 = 43 \quad \text{with} \quad d = 6 \quad \text{and} \quad y = 12.$$

Choosing now the standard magic square of order 3

8	1	6
3	5	7
4	9	2

and putting the right side of those equations, starting with -17, in that order into it, we obtain

37	-17	19
-5	13	31
7	43	-11

yielding a prime magic square with magic constant 39. For the frames we may not request that their primes are of a special form. Of course, all means of construction should be the same as in Golubev's prime magic square. Has such a magic square of primes, say of order 13, ever been constructed? Yes, one can find it in [5], and it may be republished here as a good example of magic squares of primes with no restrictions attached to their construction. It says there: "This tremendous prime magic square was sent to Francis L. Miksa of Aurora, Illinois, from an inmate in prison who, obviously, must remain nameless." The nucleus of order 3 consists of triples of primes in A.P. with $d = 6$ and $y = 3558$. Each opposite prime pair in any frame adds up to $10874 = 2 \cdot 5437$, the magic constant of order 3 is $16311 = 3 \cdot 5437$, of order 5 is $27185 = 5 \cdot 5437, \cdots$, of order 13 is $70681 = 13 \cdot 5437$. It is constructed in the same way as Golubev's magic square, but while there the difference between the largest prime, 119087, and the smallest prime, 7547, is $111540 = 2^2 \cdot 3 \cdot 5 \cdot 11 \cdot 13^2$, in the prisoner's magic square it is 9967 and 907 with $9060 = 2^2 \cdot 3 \cdot 5 \cdot 151$.

Is someone able to attach a frame of order 15 to the prisoner's remarkable magic square?

Somewhat differently behave the prime magic squares of even order. The greatest attraction is here the prime magic square of order 12 by J. N. Muncey of Jessup, Iowa, which is the smallest possible magic square of consecutive odd primes, starting with 1, ending with 827, and reproduced in [2]. It speaks for the attitude of mathematical journals

THE PRISONER'S PRIME MAGIC SQUARE

1153	8923	1093	9127	1327	9277	1063	9133	9661	1693	991	8887	8353
9967	8161	3253	2857	6823	2143	4447	8821	8713	8317	3001	3271	907
1831	8167	4093	7561	3631	3457	7573	3907	7411	3967	7333	2707	9043
9907	7687	7237	6367	4597	4723	6577	4513	4831	6451	3637	3187	967
1723	7753	2347	4603	5527	4993	5641	6073	4951	6271	8527	3121	9151
9421	2293	6763	4663	4657	9007	1861	5443	6217	6211	4111	8581	1453
2011	2683	6871	6547	5227	1873	5437	9001	5647	4327	4003	8191	8863
9403	8761	3877	4783	5851	5431	9013	1867	5023	6091	6997	2113	1471
1531	2137	7177	6673	5923	5881	5233	4801	5347	4201	3697	8737	9343
9643	2251	7027	4423	6277	6151	4297	6361	6043	4507	3847	8623	1231
1783	2311	3541	3313	7243	7417	3301	6967	3463	6907	6781	8563	9091
9787	7603	7621	8017	4051	8731	6427	2053	2161	2557	7873	2713	1087
2521	1951	9781	1747	9547	1597	9811	1741	1213	9181	9883	1987	9721

shortly before the outbreak of World War I that they would rather publish abstract mathematics than such a genuine gem of mathematical thinking. Hence, one doesn't wonder that Muncey's magic square of consecutive primes finally appeared in a philosophical journal [The Monist, 23 (1913), 623-630]. We see at a glance that this prime magic square is of a different kind. Neither has it a nucleus of order 4 nor does it include similar magic squares of order 6, 8, and 10. Its magic constant is $4514 = 2 \cdot 37 \cdot 61$.

Another gem is the magic square of order 4 consisting of 16 primes in A. P. by S. C. Root of Brookline, Massachusetts. It is published in [4]. Its magic constant is

$$15637321864 = 2^3 \cdot 43 \cdot 45457331 ,$$

the common difference is

$$223092870 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 .$$

It is not known whether there exists a sequence of 16 primes in A. P. with a smaller common difference d. Theoretically, it should be possible to find such a sequence with $d = 30030 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$.

If we have prime magic squares of an even order, the nucleus has not to consist of primes in A. P. Assuming again,

$$p_1 + d = p_2, \quad p_2 + d = p_3, \quad \cdots, \quad p_{15} + d = p_{16}$$

we shall see that the 4^{th}, the 8^{th}, and the 12^{th} d can be replaced by 6m, but these all different, say u, v, and w. Each $(2m - 1)^{\text{th}}$ d may be 30 and each $2(2m - 1)^{\text{th}}$ d may be 12. In this way we obtain the pair of prime magic squares due to the late Leo Moser of the University of Alberta which are published in [5]. Moser uses not only primes, but twin

MUNCEY'S CONSECUTIVE PRIME MAGIC SQUARE

1	823	821	809	811	797	19	29	313	31	23	37
89	83	211	79	641	631	619	709	617	53	43	739
97	227	103	107	193	557	719	727	607	139	757	281
223	653	499	197	109	113	563	479	173	761	587	157
367	379	521	383	241	467	257	263	269	167	601	599
349	359	353	647	389	331	317	311	409	307	293	449
503	523	233	337	547	397	421	17	401	271	431	433
229	491	373	487	461	251	443	463	137	439	457	283
509	199	73	541	347	191	181	569	577	571	163	593
661	101	643	239	691	701	127	131	179	613	277	151
659	673	677	683	71	67	61	47	59	743	733	41
827	3	7	5	13	11	787	769	773	419	149	751

ROOT'S MAGIC SQUARE OF PRIMES IN A. P.

2236133941	5359434121	5136341251	2905412551
4690155511	3351598291	3574691161	4020876901
3797784031	4243969771	4467062641	3128505421
4913248381	2682319681	2459226811	5582526991

MOSER'S TWIN MAGIC SQUARES OF PRIMES IN A.P.

29	1061	179	227
269	137	1019	71
1049	101	239	107
149	197	59	1091

31	1063	181	229
271	139	1021	73
1051	103	241	109
151	199	61	1093

primes. We see that $u = 6$, $v = 18$, and $w = 750$. The magic constant of the left square is $1496 = 2^3 \cdot 11 \cdot 17$, the magic constant of the right square is $1504 = 2^5 \cdot 47$. The author remembers that Leo Moser had always a little self-fabricated poem on hand which served as a kind of donkey bridge to his brain twisters: does someone recall the poem for the twin prime magic squares?

We have attempted to give a glimpse into the more recent investigations on prime magic squares and to somewhat analyze the regular ones of them. Of course, a detailed treatise on their construction would not be permissible here, but can be found in the almost classic collection of [1].

REFERENCES

1. W. S. Andrews, Magic Squares and Cubes, Dover Publications, New York, 1960.

2. Martin Gardner, "The Remarkable Lore of the Prime Numbers," Scientific American, March 1964, pp. 122-126.
3. Edgar Karst, "Magic Squares Consisting of Primes in A. P.," The Fibonacci Quarterly, April 1970, pp. 317-323.
4. Edgar Karst, "12 to 16 Primes in Arithmetical Progression," Journal of Recreational Mathematics, Oct. 1969, pp. 214-215.
5. J. S. Madachy, "Strictly for Squares," Recreational Mathematics Magazine, Oct. 1961, pp. 24-29.

[Continued from page 650.]

The determination of the branching characteristics of natural streams of class five and higher is an extremely difficult and tedious task. Thus any hypothesis proposed for stream patterns of high class is very difficult to test. If it could be shown that a Fibonacci or one of the generalized Fibonacci patterns could serve as a first approximation to natural patterns, then any hypothesis proposed could quickly and easily be explored to very high orders and the results used to plan tests that could be applied to natural patterns.

REFERENCES

1. R. E. Horton, "Erosional Development of Streams and Their Drainage Basins," Bull. Geol. Soc. Am. 56 (1945), pp. 275-370.
2. L. B. Leopold, M. G. Wolman, and J. P. Miller, Fluvial Processes in Geomorphology, W. H. Freeman, San Francisco, 1964.
3. R. B. MacArthur and J. H. Connell, The Biology of Populations, John Wiley and Sons, New York, 1966.
4. M. E. Morisawa, "Quantitative Geomorphology of Some Watersheds in the Appalachian Plateau," Bull. Geol. Soc. Am. 73 (1957), pp. 1025-1046.
5. A. E. Scheidegger, "The Algebra of Stream-Order Numbers," U. S. Geol. Survey Prof. Paper, 525B (1965), pp. 187-189.
6. A. E. Scheidegger, "Horton's Law of Stream Numbers," Water Resources Res., 4 (1968), pp. 655-658.
7. W. E. Sharp, "An Analysis of the Laws of Stream Order for Fibonacci Drainage Patterns," Water Resources Res. 7 (1971), pp. 1548-1557.
8. W. E. Sharp, "Stream Order as a Measure of Sample Source Uncertainty," Water Resources Res. 6 (1970), pp. 919-926.
9. R. L. Shreve, "Statistical Law of Stream Numbers," Jour. Geol. 74 (1966), pp. 17-37.
10. H. Steinhaus, Mathematical Snapshots, Oxford Univ. Press, New York, 1969.