ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-205 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Evaluate the determinants of n^{th} order

$$D_n = \begin{vmatrix} z & 1 & & & & \\ -1 & qz & 1 & & & \\ & -1 & q^2z & 1 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & -1 & q^{n-2}z & 1 \\ & & & & -1 & q^{n-1}z \end{vmatrix}, \quad \Delta_n = \begin{vmatrix} z & 1 & & & \\ -1 & z & q & & \\ & -1 & z & q^2 & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & -1 & z & q^{n-2} \\ & & & -1 & z \end{vmatrix} .$$

H-206 Proposed by P. Bruckman, University of Illinois, Urbana, Illinois.

Prove the identity:

$$1/(1 - x^n) = \frac{1}{n} \sum_{k=0}^{n-1} 1/(1 - x \cdot e^{2k\pi i/n}) .$$

H-207 Proposed by C. Bridger, Springfield, Illinois.

Define $G_k(x)$ by the relation

$$\frac{1}{1 - (x^2 + 1)s^2 - xs^3} = \sum_{n=0}^{\infty} G_k(x)s^k ,$$

where x is independent of s.

1. Find a recursion formula connecting the $G_k(x)$.

2. Put $x = 1$ and find $G_k(1)$ in terms of Fibonacci numbers.
3. Also with $x = 1$, show that the sum of any four consecutive G-numbers is a Lucas number.

H-208 Proposed by P. Erdos, Budapest, Hungary.

Assume

$$\frac{n!}{a_1!\,a_2!\,\cdots\,a_k!} \qquad (a_i \geq 2, \quad 1 \leq i \leq k)$$

is an integer. Show that the

$$\max \sum_{i=1}^{k} a_i < \frac{5}{2} n ,$$

where the maximum is to be taken with respect to all choices of the a_i's and k.

H-209 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Put

$$u_n = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} ,$$

where $\alpha = \beta = \alpha\beta = z$. Determine the coefficients $C(n,k)$ such that

$$z^n = \sum_{k=1}^{n} C(n,k) u_{n-k+1} \qquad (n \geq 1).$$

H-210 Proposed by G. Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

Show that a positive integer n is a Lucas number if and only if $5n^2 + 20$ or $5n^2 - 20$ is a square.

H-211 Proposed by S. Krishman, Orissa, India.

A. Show that $\binom{2n}{n}$ is of the form $2n^3k + 2$ when n is prime and $n > 3$.

B. Show that $\binom{2n-2}{n-1}$ is of the form $n^3k - 2n - n$, when n is prime.

$\binom{m}{j}$ represents the binomial coefficient, $\dfrac{m!}{j!\,(m-j)!}$.

H-212 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Let n be a positive integer. Consider n edge-connected squares. How many configurations are there if each row starts k squares to the right of the row above? (k denotes a non-negative integer.)

H-213 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

A. Let A_n be the left adjusted Pascal triangle, with n rows and columns and 0's above the main diagonal. Thus

$$A_n = \begin{pmatrix} 1 & 0 & & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 2 & 1 & 0 \cdot & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}_{n \times n}$$

Find $A_n \cdot A_n^T$ where A_n^T represents the transpose of matrix, A_n.

B. Let

$$C_n = \begin{pmatrix} 1 & 0 & 0 & & \cdots & 0 \\ 0 & 1 & 0 & & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 1 & 0 \cdot & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}_{n \times n}$$

where the i^{th} column of C_n is the i^{th} row of Pascal's triangle adjusted to the main diagonal and the other entries are 0's. Find $C_n \cdot A_n^T$.

H-214 Proposed by E. Karst, University of Arizona, Tucson, Arizona.

Let $x = y^2 + z^2$ be the first prime in a sequence of 10 primes in A. P. and

$$x + 2^2 \cdot 3^4 = (y + 2 \cdot 3^2 \cdot 7)^2 + (z - 2^5 \cdot 3^2)^2$$

the first prime in another sequence of 10 primes in A.P. where both sequences have the same common difference. The second member after the 10^{th} prime in the first sequence is divisible by 17 and has a factor which is the square of a 3-digit prime; the second member before the first prime in the second sequence is also divisible by 17, and its first three digits are a permutation of the last three digits which form a perfect square. The common difference consists of prime factors, each of them smaller than 17. Find x, y, and z.

SOLUTIONS

AN OLD FRIEND REVISITED

H-118 Proposed by G. Ledin, Jr., San Francisco, California.

Solve the difference equation

$$C_{n+2} = F_{n+2}C_{n+1} + C_n \qquad (n \geq 1)$$

with $C_1 = a$, $C_2 = b$, and F_n, the n^{th} Fibonacci number.

Solution by Clyde A. Bridger, Springfield, Illinois.

Write the following series of equations, beginning with $n = 1$,

$$\begin{aligned}
C_3 &= F_3 C_2 + a \\
C_4 &= F_4 C_3 + C_2 \\
C_5 &= F_5 C_4 + C_3 \\
&\vdots \\
C_{n+1} &= F_{n+1} C_n + C_{n-1} \\
C_{n+2} &= F_{n+2} C_{n+1} + C_n
\end{aligned}$$

We see at once that

$$C_3 = F_3 b + a = \begin{vmatrix} b & a \\ -1 & F_3 \end{vmatrix}$$

$$C_4 = F_4(F_3 b + a) + b = \begin{vmatrix} b & a & 0 \\ -1 & F_3 & 1 \\ 0 & -1 & F_4 \end{vmatrix}$$

etc. So the solution in determinant form is

$$C_{n+2} = \begin{vmatrix} b & a & 0 & 0 & \cdots & 0 & 0 \\ -1 & F_3 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & F_4 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & F_5 & \cdots & 0 & 0 \\ \cdot & & \cdot & \cdot & \cdots & & \cdot \\ \cdot & & \cdot & \cdot & \cdots & & \cdot \\ 0 & 0 & 0 & 0 & \cdots & F_{n+1} & 1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & F_{n+2} \end{vmatrix}$$

as may be verified by expanding in terms of the minors of the last row.

The ratio of two adjacent solutions of the difference equation can be developed into a continued fraction. Write, using the above sets of equations,

$$\begin{aligned}
\frac{C_3}{C_2} &= F_3 + \frac{a}{b} \\
\frac{C_4}{C_3} &= F_4 + \frac{1}{C_3/C_2} = F_4 + \cfrac{1}{F_3 + \frac{a}{b}} \\
&\vdots \\
\frac{C_{n+2}}{C_{n+1}} &= F_{n+2} + \cfrac{1}{F_{n+1} + \cfrac{1}{F_n + \cfrac{\ddots}{\cfrac{1}{F_3 + \frac{a}{b}}}}} \;.
\end{aligned}$$

Also solved by R. Whitney.

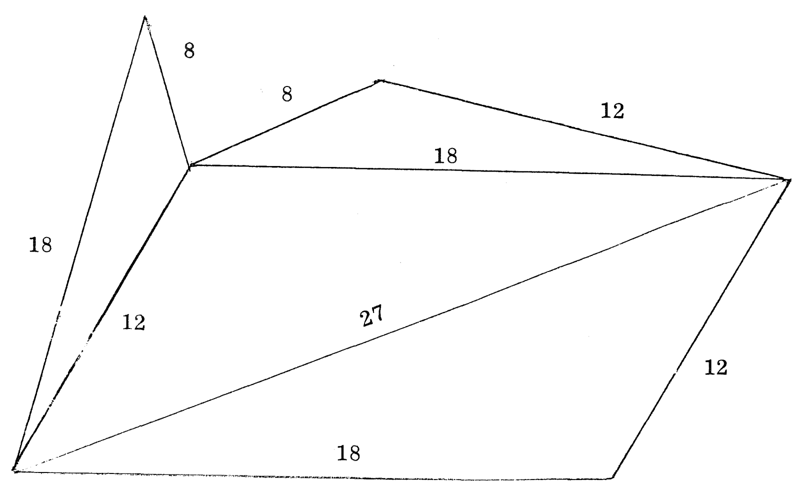
ANOTHER OLD TIMER

H-108 Proposed by H. E. Huntley, Hutton, Somerset, U.K.

Find the sides of a tetrahedron, the faces of which are all scalene triangles similar to each other, and having sides of integral lengths.

Solution by the Proposer.

The interesting article, "Mystery Puzzle and Phi," by Marvin H. Holt (Fibonacci Quarterly, Vol. 3, No. 2, p. 135) contains a solution. See H. E. Huntley's The Divine Proportion, Dover, New York, N. Y., 1970, pp. 108-109, Section entitled "The Tetrahedron Problem."



SHADES OF THE PAST

H-86 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, Calif. (Corrected)

Let p, q be integers such that $p + q \geq 1$, $q \geq 0$; show that if $x^p(x-1)^q - 1 = 0$ has roots $r_1, r_2, \cdots, r_{p+q}$ and $(x-1)^{p+q} - x^p = 0$ has roots $s_1, s_2, \cdots, s_{p+q}$ then $s_i^q = r_i^{q+p}$ for $i = 1, 2, \cdots, p+q$.

Solution by L. Carlitz, Duke University, Durham, North Carolina.

Presumably the problem should read:

Show that if $x^p(x-1)^q - 1 = 0$ has roots $r_1, r_2, \cdots, r_{p+q}$ and $(y-1)^{p+q} - y^p = 0$ has roots $s_1, s_2, \cdots, s_{p+q}$, then the roots can be so numbered that

$$r_i^{p+q} = s_i^q \qquad (i = 1, 2, \cdots, p+q) .$$

Proof. Consider the transformation

$$x - 1 = \frac{1}{y - 1} .$$

This implies

$$y = \frac{x}{x - 1} .$$

Hence, if x satisfies $x^p(x - 1)^q = 1$, we get

$$y^q = \frac{x^q}{(x - 1)^q} = \frac{x^{p+q}}{x^p(x - 1)^q} = x^{p+q} .$$

This evidently yields the stated result.

PARTIAL SOLUTION

H-125 Proposed by Stanley Rabinowitz, Far Rockaway, New York.

Define a sequence of positive integers to be left-normal if given any string of digits, there exists a member of the given sequence beginning with this string of digits, and define the sequence to be right-normal if there exists a member of the sequence ending with this string of digits.

Show that the sequences whose n^{th} terms are given by the following are left-normal but not right-normal.

a. $P(n)$, where $P(x)$ is a polynomial function with integral coefficients.

b. P_n, where P_n is the n^{th} prime.

c. $n!$

d. F_n, where F_n is the n^{th} Fibonacci number.

Partial Solution by R. Whitney, Lock Haven State College, Lock Haven, Pennsylvania.

b. The article "Initial Digits for the Sequence of Primes," by R. E. Whitney (Amer. Math. Monthly, Vol. 79, No. 2, 1972, pp. 150-152) established a positive relative logarithmic density for the set of primes with initial digit sequence $\{a_n, a_{n-1}, \cdots, a_1\}$ in the set of primes. Thus P_n is left-normal. On the other hand, no member of P_n ends in "4," so P_n is not right-normal.

I believe that the left-normality of F_n can also be established with a density argument.

Editorial Note

The following list represents those problems for which no solutions have been submitted. Let's fight problem pollution!

H-76, H-84, H-87, H-90, H-91, H-84, H-100, H-110, H-113, H-114, H-115, H-116, H-122, H-125 (partial), H-130, H-146, H-148, H-152, H-170, H-174, H-179, H-182. This list represents problems less than or equal to H-185.