A CURIOUS PROPERTY OF UNIT FRACTIONS OF THE FORM 1/d WHERE (d, 10) = 1

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INTRODUCTION

One of the rewards of teaching is seeing your students discover for themselves a profound mathematical result. Over twelve years of teaching I have had more than my share of such observances. Perhaps the most rewarding came about as a "spin off" of a problem dealing with the nature of a repeating decimal. A student at San Carlos High School, Frank Stroshane, made the original discovery described in this article while trying to find out why with some fractions its period has a "nines-complement" split, that is, its period can be split into two halves that have a nines complement relationship. For example: $1/7 = .\overline{142857}$ has 1 and 8; 4 and 5; and 2 and 7. Frank, typical of talented students, found a different "gem." He could not prove his result but it was clear to me and the others with whom he shared it that it was unquestionably true. It is this observation and its subsequent justification that represents the main thrust of the article.

The property alluded to is:

<u>Theorem</u>. The period of a fraction of the form $1/d$ where $(d,10) = 1$ can be completely determined without dividing.

For example, to find the decimal expansion of $1/7$ we "know" (this knowledge will be proven later) that the last digit in the period must be 7. Now, multiply this terminal digit by our "magic" number 5 (this too will be explained later). Continue the process of multiplying the previous digit by 5 (allow for carries) until the digits of the period repeat. The full process follows:

a. $1/7$ has 7 for its last digit and its period	7
b. Multiply the 7 by 5 giving	$\overset{3}{}57$ where 3 is the carry
c. Multiply the 5 by 5 and add the previous carry of 3 giving	$\overset{2}{}857$ where 2 is the carry
d. Repeating the process gives	$\overset{4}{}2857$
e. Again giving	$\overset{1}{}42857$

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[1]Provided the proof of the algorithm described in the article.
[2]A teacher at San Carlos High School. Presented the problem that led, after several years, to the proof. He also compiled this article.
[3]A student at San Carlos High School. Discovered the magic number and provided most of the lemmas and their proofs leading to Brother Brousseau's proof.

f. Again giving $\begin{array}{l} 2 \\ \;142857 \end{array}$

g. Again giving $\begin{array}{l} 0 \\ 7142857 \end{array}$

which indicates the period is repeating.

Therefore $1/7 = .\overline{142857}$

Before launching into a statement of the algorithm employed and its proof, some preliminaries need to be established. We have assumed that all fractions of the form $1/d$ where $(d,10) = 1$ have a decimal expansion which repeats, furthermore they begin their period immediately.* First a Lemma about the final digit in the repeated block.

Lemma. If $1/d = .\overline{a_1 a_2 \cdots a_k}$ where $(d,10) = 1$, then

$$d \cdot a_k = 9 \mod 10$$

or

$$d \cdot a_k \text{ ends in a } 9 .$$

Proof. Since

$$1/d = .\overline{a_1 a_2 \cdots a_k}$$

then

$$\frac{10^k}{d} = a_1 a_2 \cdots a_k \cdot \overline{a_1 a_2 \cdots a_k}$$

subtracting

$$\frac{10^k - 1}{d} = a_1 a_2 \cdots a_k$$

or

$$\frac{10^k - 1}{d} = a_1 \cdot 10^{k-1} + a_2 \cdot 10^{k-2} + \cdots + a_{k-1} 10^1 + a_k 10^0$$

$$10^k - 1 = d(a_1 \cdot 10^{k-1} + a_2 \cdot 10^{k-2} + \cdots + a_{k-1} \cdot 10^1 + a_k \cdot 10^0)$$

but

$$10^k - 1 \equiv 9 \pmod{10}$$

$$d(a_1 \cdot 10^{k-1} + a_2 \cdot 10^{k-2} + \cdots + a_{k-1} 10) + d \cdot a_k \equiv 9 \pmod{10}$$

or

$$d \cdot a_k \equiv 9 \pmod{10}$$

since

$$d(a_1 \cdot 10^{k-1} + a_2 \cdot 10^{k-2} + \cdots + a_{k-1} \cdot 10) \equiv 0 \pmod{10}$$

or

$$d \cdot a_k = 10N + 9$$

where N is some integer; that is, $d \cdot a_k$ ends in a 9.

This shows that in expanding any unit fraction of the type described, the product of the denominator and the last digit must end in a nine. Hence $1/7$ has for the last digit in its

*See The Enjoyment of Mathematics, Rademacher and Toeplitz, pp. 149-152.

period a 7; 1/11 has for the last digit in its period a 9; and 1/23 has for the last digit in its period a 3.

FINDING THE "MAGIC NUMBER"

An algorithm for determining the magic number is as follows:

1. Find the terminal digit in the period (see Lemma).
2. Multiply by d.
3. Add 1.
4. Drop final digit in sum. (It will always be zero.)
5. This number will be the "magic number."

Briefly, if m is the "magic number,"

$$m = \frac{d \cdot a_k + 1}{10} ,$$

where d is the denominator of the given unit fraction, a_k is the terminal digit in the period of 1/d, and k is length of period. Therefore, using the above algorithm, the "magic number" for the following unit fractions are:

a. For 1/7 the magic number is 5, since

$$5 = \frac{7(7) + 1}{10} .$$

b. For 1/11 the magic number is 10, since

$$10 = \frac{9(11) + 1}{10} .$$

c. For 1/27 the magic number is 19, since

$$19 = \frac{7(27) + 1}{10} .$$

d. For 1/43 the magic number is 13, since

$$13 = \frac{3(43) + 1}{10} ,$$

etc.

PROOF OF ALGORITHM

On inspection one can see this algorithm is equivalent to finding the quantity which on being multiplied by 10 and divided by the denominator gives a remainder of 1. That is,

$$10m \equiv 1 \pmod{d} .$$

If we visualize the process of division in complete detail, m is the remainder in the division process just prior to the remainder 1 which initiates a new cycle.

How does one go about justifying such an algorithm? First, it may be pointed out that the length of the period of such a decimal is found by the smallest value of k for which

$$10^k = 1 \pmod{d} ,$$

where d is an odd integer.* Thus for 7

$$10^1 \equiv 3 \pmod{7}; \quad 10^2 \equiv 2 \pmod{7}; \quad 10^3 \equiv 6 \pmod{7};$$
$$10^4 \equiv 4 \pmod{7}; \quad 10^5 \equiv 5 \pmod{7}; \quad 10^6 \equiv 1 \pmod{7} .$$

Note also that these quantities are the successive remainders in the division process. The magic number is given by $10^5 \equiv 5 \pmod{7}$. In other words, the magic number m is the least positive residue for which $10^{k-1} \equiv m \pmod{d}$. It is also the last remainder in the division process that precedes a remainder of 1 which is the first remainder. That is

$$10^{k-1} = r_k \pmod{d} ,$$

where r_k is the last remainder where the length of period is k.

To understand the ensuing analysis, let us parallel division by 7 and the corresponding notation that will be employed.

$$\begin{array}{r} .142857 \\ 7\,\overline{)\,1.000000} \\ \underline{7} \\ 30 \\ \underline{28} \\ 20 \\ \underline{14} \\ 60 \\ \underline{56} \\ 40 \\ \underline{35} \\ 50 \\ \underline{49} \\ 1 \end{array} \qquad\qquad \begin{array}{l} .a_1 a_2 a_3 a_4 a_5 a_6 \\ d\,\overline{)\,1.0\,0\,0\,0\,0\,0} \\ \underline{n_1} \\ r_2 0 \\ \underline{n_2} \\ r_3 0 \\ \underline{n_3} \\ r_4 0 \\ \underline{n_4} \\ r_5 0 \\ \underline{n_5} \\ r_6 0 \\ \underline{n_6} \\ 1 = r \end{array}$$

*The proof of this statement can be found in The Enjoyment of Mathematics.

In the above illustration, $n_2 = a_2 \cdot d$, while $r_3 0$ is the remainder with a zero attached. From the nature of the division operation we have the following equations:

$$10\, r_1 = a_1 \cdot d + r_2$$

$$10\, r_2 = a_2 \cdot d + r_3$$

$$\cdots$$

$$10\, r_{k-1} = a \cdot d + r_k$$

$$10\, r_k = a_k \cdot d + 1 \quad .$$

Taking

$$r_1 = 1; \qquad 10 \cdot r_1 = a_1 \cdot d + r_2$$

implies

$$10 = a_1 \cdot d + r_2 \quad \text{or} \quad 10 - a_1 \cdot d = r_2$$

and

$$10 \cdot r_2 = a_2 \cdot d + r_3$$

leads to

$$10 \cdot (10 - a_1 \cdot d) = a_2 \cdot d + r_3$$

or

$$10^2 - r_3 = (10a_1 + a_2)\, d$$

or equivalently

$$10^2 \equiv r_3 \pmod{d}$$

and

$$10 \cdot r_3 = a_3 \cdot d + r_4$$

leads to

$$10(10^2 - 10\, a_1 \cdot d - a_2 \cdot d) = a_3 \cdot d + r_4$$

or

$$10^3 - r_4 = (10^2\, a_1 + 10\, a_2 + a_3)d$$

or

$$10^3 \equiv r_4 \pmod{d}$$

and in general

$$10^{\ell} \equiv r_{\ell+1} \pmod{d} \quad .$$

Now since $r_k \equiv 10^{k-1} \pmod{d}$ where r_k is the last remainder in the division process for the unit fraction which has a decimal expansion with a period of length k it follows (recalling $10^k \equiv 1 \pmod{d}$),

$$r_k^2 \equiv (10^{k-1})^2 \equiv 10^{2k-2} \equiv 10^{k-2} \equiv r_{k-1} \pmod{d}$$

or equivalently

$$r_{k-1} \equiv 10^{k-2} \pmod{d}$$

so that

$$r_k^2 = d \cdot b_k + r_{k-1} ,$$

where b_k is an integer. Therefore

$$r_k \cdot r_{k-1} \equiv 10^{k-1} 10^{k-2} \equiv 10^{2k-2} \equiv 10^{k-3} \equiv r_{k-2} \pmod{d} .$$

In general,

$$r_k \cdot r_{k-\lambda} = 10^{k-1}10^{k-\lambda-1} = 10^{2k-\lambda-2} = 10^{k-\lambda-2} = r_{k-\lambda-1} \pmod{d} .$$

Hence

$$r_k^2 = d \cdot b_k + r_{k-1}$$
$$r_k \cdot r_{k-1} = d \cdot b_{k-1} + r_{k-2}$$
$$r_k \cdot r_{k-2} = d \cdot b_{k-2} + r_{k-3}$$
$$\cdot \quad \cdot \quad \cdot$$
$$r_k \cdot r_2 = d \cdot b_2 + 1$$

where the b_i's are integers.

From the first set of relations,

$$a_k d = 10\, r_k - 1$$
$$r_k a_k d = 10\, r_k^2 - r_k = 10\, d \cdot b_k + 10\, r_{k-1} - r_k$$
$$= 10\, d \cdot b_k + a_{k-1} d$$

therefore

$$r_k a_k = 10\, b_k + a_{k-1} .$$

This shows that the product of a magic number r_k by the last digit in the period a gives the penultimate digit in the period, viz, a_{k-1}. Continuing in like manner:

$$a_{k-1} d = 10\, r_{k-1} - r_k$$
$$r_k a_{k-1} d = 10\, r_k r_{k-1} - r_k^2$$
$$= 10\, d \cdot b_{k-1} + 10\, r_{k-2} - d \cdot b_k - r_{k-1}$$
$$= 10\, d \cdot b_{k-1} + a_{k-2} d - d b_k$$

since $10\, r_{k-2} - r_{k-1} = a_{k-2} d$ or simplifying,

$$r_k a_{k-1} = 10\, b_{k-1} + a_{k-2} - b_k$$
$$r_k a_{k-1} + b_k = 10\, b_{k-1} + a_{k-2} .$$

This shows that multiplying r_k by a_{k-1}, the next to last digit in the period and adding b_k from previous operation gives a_{k-2} as the last digit. In general,

$$a_{k-\lambda} d = 10\, r_{k-\lambda} - r_{k-\lambda+1}$$
$$r_k a_{k-\lambda} d = 10\, r_k \cdot r_{k-\lambda} - r_k \cdot r_{k-\lambda+1}$$
$$= 10\, d \cdot b_{k-\lambda} + 10\, r_{k-\lambda-1} - d b_{k-\lambda+1} - r_{k-\lambda}$$
$$= 10\, d b_{k-\lambda} - d b_{k-\lambda+1} + d \cdot a_{k-\lambda-1} ,$$

since $10\, r_{k-\lambda-1} - r_{k-\lambda} = d \cdot a_{k-\lambda-1}$ or

$$r_k \cdot a_{k-\lambda} + b_{k-\lambda+1} = 10\, b_{k-\lambda} + a_{k-\lambda-1} .$$

This shows that the process continues at each step of the operation and completes the proof.

FINAL COMMENTS

It is not difficult to expand the remarks concerning unit fractions developed in this article to all fractions of the form c/d where $0 < c < d$ and $(d, 10) = 1$. Also the fact that the remainders in the division process are all relatively prime to the division is useful in determining the length of the period of a given fraction. A proof of this result concludes the article.

Theorem. All of the remainders in the division process associated with $1/d$ where $(d, 10) = 1$ are relatively prime to d.

Proof. Since $r_1 = 1$ then $(r, d) = 1$.

$$10\,r_1 = a_1 \cdot d + r_2 \qquad (0 \le r_2 < d)\ .$$

It must be that $(r_2, d) = 1$ since if

$$(r_2, d) = t_1\ ; \qquad (t_1 \ne 1)$$

then

$$r_2 = pt_1 \quad \text{and} \quad d = kt_1\ .$$

$$10 = a_1(kt_1) + pt_1 = (a_1 \cdot k + p)t_1\ .$$

Therefore, t_1 must divide 10 but $(d, 10) = 1$ and $d = kt_1$ hence a contradiction and $(r_2, d) = 1$. Continuing,

$$10\,r_2 = a_2 \cdot d + r_3 \qquad (0 \le r_3 < d)\ .$$

Again it must be that $(r_3, d) = 1$ since if

$$(r_3, d) = t_2 \qquad (t_2 \ne 1)$$

then

$$r_3 = pt_2 \quad \text{and} \quad d = pt_2$$

$$10\,r_1 = a_2 \cdot k \cdot t_2 + pt_2 = (a_2 \cdot k + p)t_2$$

but $(t_2, r_1) = 1$ since $(d, r_1) = 1$ hence t_2 must divide 10 but $(d, 10) = 1$ thus $(r_2, d) = 1$. Since the argument continues in like manner, the theorem is proved.

EDITORIAL COMMENT
Marjorie Bicknell

Puzzles intimately related to the results of the paper, "A Curious Property of Unit Fractions of the Form $1/d$ Where $(d, 1) = 1$," have the following form:

Find a number whose left-most digit is k which gives a number $1/m$ as large when k is shifted to the far right-end of the number.

The solution to such puzzles can be obtained by multiplying k by the "magic multiplier" m to produce the original number, which is the repeating block of the period of c/d, where m is the "magic number" for d, and $1 < c < d$.

For example, find a number whose left-most digit is 6 which gives a number $1/4$ as large when 6 is shifted to the far right end of the number. Multiplying 6 by the "magic multiplier" 4 as explained in the paper above gives a solution of 615384, which is four times as great as 153846. Notice that

$$4 = \frac{d \cdot a_k + 1}{10}$$

gives the solutions, in positive integers,

$$d \cdot a_k = 39 = 39 \cdot 1 = 13 \cdot 3 ,$$

where $d = 13$ or $d = 39$ give the same solution as follows. $1/13$ ends in 3, $2/13$ ends in 6,

$$4 \times \frac{2}{13} = \frac{8}{15}$$

has the original number of the puzzle as its period.

As a second example, re-read the puzzle using $k = 4$ and $m = 2$. Multiplying 4 using the "magic multiplier" 2 yields

421052631578947368

which is twice as large as

210526315789473684 .

Here 2 in the "magic number" formula produces

$$2 = (d \cdot a_k + 1)/10$$

so that

$$d \cdot a_k = 19 = 19 \cdot 1 .$$

$1/19$ ends in 1, $4/19$ ends in 4, $2 \cdot 4/19 = 8/19$ which has the original number as its period. ($14/19$ also ends in 4 but $2 \cdot 14 > 19$.)

One can also find $1/m$, $(m, 10) \neq 1$ by methods of this paper. $1/6 = (1/2)(1/3)$. Find $1/3 = .3333 \cdots$ without dividing. Then $(.5) \times (.3333 \cdots)$, remembering that the multiplication on the right begins with 1 to carry, makes $.1666 \cdots$.