BERNOULLI NUMBERS AND NON-STANDARD DIFFERENTIABLE STRUCTURES ON (4k - 1) - SPHERES

HELAMAN ROLFE PRATT FERGUSON*

Department of Mathematics, University of Washington, Seattle, Washington

ABSTRACT

A number theoretical conjecture of Milnor is presented, examined and the existence of non-standard differentiable structures on $(4k - 1)$-spheres for integers k, $4 \le k \le 265$, is proved.

1. INTRODUCTION

In 1959, J. Milnor [1] proved the following theorem concerning non-standard differentiable structures on $(4k - 1)$-spheres.

<u>Theorem 1.</u> If r is an integer, such that $k/3 < r \le k/2$, then there exists a differentiable manifold M, homeomorphic to S^{4k-1} with $\lambda(M) \equiv s_r s_{k-r} N/s_k \pmod 1$, where $s_k = 2^{2k}(2^{2k-1} - 1)B_k/(2k)!$, all of the prime factors of the integer N are less than $2(k - r)$, B_k is the k^{th} Bernoulli number in the sequence $B_1 = 1/6$, $B_2 = 1/30$, $B_2 = 1/42$, $B_4 = 1/30, \cdots$, and λ is an invariant associated with the manifold M.

Milnor presents an algorithm based on Theorem 1, proves structures exist for $k = 2$, 4, 5, 6, 7, 8, conjectures that Theorem 1 implies the existence of these structures for $k > 3$, and states that he has verified the conjecture for $k < 15$. He points out that for $k = 1$ and $k = 3$ no integers r exist in the interval $(k/3, k/2]$ and that for $k = 1$, two differentiable homeomorphic 3-manifolds are diffeomorphic.

The Milnor algorithm will be described by considering the first seven cases. In each case an actual lower bound will be calculated for the number of said structures; to calculate this bound we consider the denominator of the reduced fraction and drop all prime factors less than $2(k - r)$.

1. $k = r$, $r = 2$.

$$\binom{8}{4}(2^3 - 1)^2 B_2^2/(2^7 - 1)B_4 = (7^3/3)(1/127), \qquad 1b = 127\,.$$

2. $k = 6$, $r = 3$.

$$\binom{10}{4}(2^3 - 1)(2^5 - 1)B_2 B_3/(2^9 - 1)B_5 = (11/5)(31/73), \qquad 1b = 73\,.$$

*Research supported in part by an NSF Summer Teaching Fellow Grant, also by NSF grant GP-13708, and by the BYU Computer Center (for 20 consecutive hours of computation time!).

Copies of the tables referred to in the text may be obtained from the writer at the address listed in the current Combined Membership List of the AMS.

3. $k = 6, \quad r = 3.$

$$\binom{12}{6}(2^5 - 1)^2 B_3^2 / (2^{11} - 1)B_6 = (2\cdot 5\cdot 11\cdot 13)(31^2/23\cdot 89\cdot 691) \ ,$$

$$1b = 23\cdot 89\cdot 691 \ .$$

4. $k = 7, \quad r = 3.$

$$\binom{14}{6}(2^5 - 1)(2^7 - 1)B_3 B_4 / (2^{13} - 1)B_7 = (11\cdot 13/2\cdot 5\cdot 7)(31\cdot 127/8191) \ ,$$

$$1b = 8191 \ .$$

5. $k = 8, \quad r = 3.$

$$\binom{16}{6}(2^5 - 1)(2^9 - 1)B_3 B_5 / (2^{15} - 1)B_8 = (2^2\cdot 5^2\cdot 13\cdot 17/3)(73/151\cdot 3617) \ ,$$

$$1b = 151\cdot 3617 \ .$$

6. $k = 9, \quad r = 4.$

$$\binom{18}{8}(2^7 - 1)(2^9 - 1)B_4 B_5 / (2^{17} - 1)B_9 = (2\cdot 3\cdot 7^2\cdot 13\cdot 17\cdot 19)/(73\cdot 127/43867\cdot 131071) \ ,$$

$$1b = 43867\cdot 131071 \ .$$

7. $k = 10, \quad r = 4 \ .$

$$\binom{20}{8}(2^7 - 1)(2^{11} - 1)B_4 B_6 / (2^{19} - 1)B_{10} = (11\cdot 17\cdot 19/7)(23\cdot 89\cdot 127/283\cdot 617\cdot 524287) \ ,$$

$$1b = 283\cdot 617\cdot 524287 \ .$$

8. $k = 10, \quad r = 4 \ .$

$$\binom{20}{10}(2^9 - 1)^2 B_5^2 / (2^{19} - 1)B_{10} = (2\cdot 5^3\cdot 7^2\cdot 13\cdot 17\cdot 19/3)(73^2/283\cdot 617\cdot 524287) \ ,$$

$$1b = 283\cdot 617\cdot 524287 \ .$$

9. $k = 8, \quad r = 4 \ .$

$$\binom{16}{8}(2^7 - 1)^2 B_4^2 / (2^{15} - 1)B_8 = (3\cdot 5\cdot 11\cdot 13\cdot 17/7)(127^2/31\cdot 151\cdot 3617) \ ,$$

$$1b = 31\cdot 151\cdot 3617 \ .$$

There will be $[k/2] - [k/3]$ integers in the interval $(k/3, k/2]$ and one may choose the largest of the lower bounds. We now restate the positive outcome of the algorithm in the form of the following

Conjecture 1. Let r be an integer, $r \in (k/3, k/2]$, $k > 3$,

$$\binom{2k}{2k}(2^{2r-1} - 1)(2^{2k-2r-1} - 1)B_r B_{k-r}/(2^{2k-1} - 1)B_k = a/b, \qquad (a, b) = 1,$$

then there exists a prime number p, $p > 2(k - r)$, such that p divides b.

This purely number theoretic conjecture implies the existence of more than $2(k - r)$ non-standard differentiable structures for S^{4k-1}, the $(4k - 1)$-dimensional sphere. Conjecture 1 has, aside from its aesthetic number theoretical interest, the additional significance of important topological consequences, and is one more example of the ubiquitous nature of the Bernoulli numbers.

2. REPRESENTATION STRUCTURE OF THE BERNOULLI NUMBERS

Although the Bernoulli numbers have been objects of published mathematical thought for over two centuries, in some respects, embarrassingly little is known about them. We shall present the features of these numbers useful to us in examining Conjecture 1.

As a typical beginning point we write [2]

$$x(e^x - 1) = \sum_{k=0}^{\infty} b_k x^k/k! \tag{1}$$

and since $b_0 = 1$, $b_1 = -1/2$, and $x/(e^x - 1) + x/2$ is an even function, we write

$$b_{2k} = (-1)^{k-1}B_k \quad \text{and} \quad b_{2k+1} = 0, \qquad k \geq 1.$$

We have

$$1 - (1/2)\cot(x/2) = \sum_{k=1}^{\infty} B_k x^{2k}/(2k)! \tag{2}$$

and by the double series theorem [3], we see that

$$B_k = 2(2k)!\,\zeta(2k)/(2\pi)^{2k}, \tag{3}$$

where

$$\zeta(2k) = \sum_{n=1}^{\infty} n^{-2k},$$

the Dirichlet series usually referred to as the even zeta function. An equivalent definition to (1) is the umbral recursion [4].

$$(b+1)^k - b_k = 0, \qquad b_0 = 1, \tag{4}$$

which reduces to

$$\sum_{r=0}^{k} \binom{k+1}{r} b_r = 0, \qquad b_0 = 1. \tag{5}$$

Equation (1) is the reciprocal of

$$\sum_{k=0}^{\infty} x^k/(k+1)!$$

and an expression for the b_k may be written with symmetric functions of the coefficients of the reciprocal of (1). We may rather write [5], [6]

$$x/(e^x - 1) = \sum_{m=0}^{\infty} (-1)^m \left(\sum_{k=1}^{\infty} x^k/(k+1)! \right)^m \tag{6}$$

so that [7]

$$B_k = (-)^{k-1} \sum_{m=1}^{2k} (-)^m \sum \binom{m}{a_1, \cdots, a_{2k}} \binom{2k}{(1;a_1), \cdots, (2k;a_{2k})} \times (1/2^{a_1} \cdot 3^{a_2} \cdots (2k+1)^{a_{2k}} \tag{7}$$

where the sum is over the partitions of

$$2k, \quad \sum_{i=1}^{2k} a_i = m, \quad \sum_{i=1}^{2k} i a_i = 2k,$$

$$\binom{m}{a, b, c, \cdots} = m!/a!b!c! \cdots,$$

$$\binom{m}{(a;\alpha), \cdots (d;\beta)} = m!/(a!)^{\alpha} \cdots (d!)^{\beta},$$

and there will be p(2k) terms [8]. A variant of (7) is

$$(-)^{k-1}B_k = -(1/2k+1) + \sum (-)^m \prod_{p<2k} p^{\delta(p,k,a_1,\cdots,a_{2k})} \tag{8}$$

where the product is over all prime numbers less than $2k$, the functions $\delta(p, k, a_1, \cdots, a_{2k})$ are all integers and the sum is over all the partitions of $2k$ but one.

The calculation of Bernoulli numbers has been a lively subject [9], and there exist several tables of these numbers. [The most massive is D. Knuty, MTAC, Unpublished Mathematical Tables File. The caretaker of this file, J. W. Wrench, has informed us that from Knuth's manuscript of 1270D values of $10^{-8k}B_k$ for $k = 1(1)250$ one can obtain the exact values of only the first 159 Bernoulli numbers.] To facilitate the computation of Bernoulli and related numbers, Lehmer generalized a process of Kronecker to produce lacunary recurrences of which the following are typical [10].

$$\sum_{\lambda=0}^{[m/2]} (-)^{\lambda} 2^{m-2\lambda} B_{m-2\lambda} \binom{2m+2}{2\lambda+2} = (-)^{[m/2]}(m+1)/2, \tag{9}$$

$$\sum_{\lambda=0}^{[m/2]} B_{m-2\lambda} \binom{2m+4}{4\lambda+4} \left((-)^{\lambda} 2^{2\lambda+1} + 1\right) = \left((m+2)/2\right)\left((-)^{[m/2]} 2^{m+1} + 1\right), \tag{10}$$

$$\sum_{\lambda=0}^{[m/3]} B_{m-3\lambda} \binom{2m+3}{6\lambda+3} = \begin{cases} -(2m+3)/6, & \text{if } m = 3k-1, \\ (2m+3)/3, & \text{otherwise}, \end{cases} \tag{11}$$

$$\sum_{\lambda=0}^{[m/4]} B_{m-4\lambda} \binom{2m+4}{8\lambda+4} 2^{m+1-2[(m+1)/4]-2\lambda} \mathfrak{M}_{4\lambda+2} = (-)^{[m/2]}(m+2)\mathfrak{M}_{m+2} \tag{12}$$

where

$$\mathfrak{M}_n = -34\mathfrak{M}_{n-4} - \mathfrak{M}_{n-8} \quad \text{and} \quad \mathfrak{M}_n = 2, 0, 3, 10, 14, -12, -99, -338,$$

for $n = 0, 1, 2, 3, 4, 5, 6, 7,$ respectively.

(13)

$$\sum_{\lambda=0}^{[m/6]} B_{m-6\lambda} \binom{2m+6}{12\lambda+6} \left(\mathfrak{B}_{6\lambda+2} + (-)^{\lambda} 2^{6\lambda+2}\right) = \begin{cases} \left((m+3)/3\right)\left(\mathfrak{B}_{m+2} + (-)^{[m/2]} 2^{m+2}\right), \\ \quad \text{if } m \not\equiv 2(3); \end{cases}$$

or
$$\begin{cases} -\big((m+3)/6\big)\big(\mathfrak{B}_{m+2} + (-)^{[m/2]}2^{m+2} - (-)^{((m+1)/3)}3\big), \\ \qquad \text{if } m \equiv 2(3), \end{cases}$$

where

$$\mathfrak{B}_n = -2702\mathfrak{B}_{n-6} - \mathfrak{B}_{n-12},$$

and

$$\mathfrak{B}_n = 1, 5, 26, 97, 265, 362, -1351, -13775, -70226, -262087, -716035, -978122,$$

for $n = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$, respectively.

The point of creating lacunary recurrences is to avoid dealing with all the B_r, say $r < k$, to calculate B_k. An example of a recursion relation which is not precisely lacunary yet satisfies this last condition is

(14)
$$B_k = (k/2)\binom{2k-2}{k-1} + k\binom{2k}{k}\sum_{r=0}^{[k/2]}(-)^r B_r\binom{k}{2k}\big(1/(2k-2r)\big) + \sum_{0\le r,s\le[k/2]} B_r B_s \times \binom{2k}{2r,\ 2s,\ 2k-2r,\ 2k-2s}\big(1/(2k-2r-2s-1)\big),$$

which can be proved [11] by repeated integration of the Fourier series for $(\pi - x)/2$ and then using Parseval's Theorem on the result.

From (2) above, we have the identity

$$(d/dx)\Big(x\big(1-(x/2)\cot(x/2)\big)\Big) = x^2/4 + \big(1-(x/2)\cot(x/2)\big)^2. \tag{15}$$

Hence, we extract

$$(2k+1)B_k = \sum_{r=1}^{[k/2]} 2^{g(r)}\binom{2k}{2r}B_r B_{k-r}, \tag{16}$$

where

$$g(r) = \begin{cases} 1 & \text{if } r < [k/2] \text{ or } r = [k/2],\ k \text{ odd}, \\ 0 & \text{if } r = [k/2],\ k \text{ even}. \end{cases}$$

We observe that this "quasi-convolution" recurrence involves only positive numbers; hence, beginning with

$$B_1 = 1/2\cdot 3, \tag{17}$$

$$B_2 = 1/2\cdot 3\cdot 5, \tag{18}$$

(19) $$B_3 = 1/2\cdot 3\cdot 7\,,$$

(20) $$B_4 = (1/2\cdot 3^4\cdot 5)(2^2\cdot 5 + 7) = 1/2\cdot 3\cdot 5\,,$$

(21) $$B_5 = (1/2\cdot 3^3\cdot 11)(2^2\cdot 5 + 7 + 2\cdot 3^2) = (5/2\cdot 3\cdot 11)\,,$$

(22) $$\begin{aligned} B_6 = {} & (1/2\cdot 3^3\cdot 5\cdot 7\cdot 13)(2^3\cdot 5^2\cdot 7 + 2\cdot 5\cdot 7^2 + 2^2\cdot 5\cdot 7\cdot 11 + 7^2\cdot 11 + 2^2\cdot 3^2\cdot 5\cdot 7 \\ & + 2\cdot 3^2\cdot 5\cdot 11) = 691/(2\cdot 3\cdot 5\cdot 7\cdot 13)\,, \end{aligned}$$

(23) $$\begin{aligned} B_7 = {} & (1/2\cdot 3^5\cdot 5^2)(2^3\cdot 5^2\cdot 7 + 2\cdot 5\cdot 7^2 + 2^2\cdot 3^2\cdot 5\cdot 7 + 2^2\cdot 5\cdot 7\cdot 11 \\ & + 7^2\cdot 11 + 2\cdot 3^2\cdot 5\cdot 11 + 2^2\cdot 5\cdot 7\cdot 13 + 7^2\cdot 13 + 2\cdot 3^2\cdot 7\cdot 13 \\ & + 2^2\cdot 5\cdot 11\cdot 13 + 7\cdot 11\cdot 13) = 7/(2\cdot 3)\,, \end{aligned}$$

(24) $$\begin{aligned} B_8 = {} & (1/2\cdot 3^2\cdot 5\cdot 17)(2^5\cdot 3\cdot 5^2\cdot 7 + 2^3\cdot 3\cdot 5\cdot 7^2 + 2^4\cdot 3^3\cdot 5\cdot 7 \\ & + 2^4\cdot 3\cdot 5\cdot 7\cdot 11 + 2^2\cdot 3\cdot 7^2\cdot 11 + 2^3\cdot 3^3\cdot 5\cdot 11 + 2^4\cdot 3\cdot 5\cdot 7\cdot 13 \\ & + 2^2\cdot 3\cdot 7^2\cdot 13 + 2^3\cdot 3^3\cdot 7\cdot 13 + 2^4\cdot 3\cdot 5\cdot 11\cdot 13 + 2^2\cdot 3\cdot 7\cdot 11\cdot 13 \\ & + 2^5\cdot 3^2\cdot 5^2\cdot 7 + 2^3\cdot 3^2\cdot 5\cdot 7^2 + 2^4\cdot 3^4\cdot 5\cdot 7 + 2^4\cdot 3^2\cdot 5\cdot 7\cdot 11 \\ & + 2^2\cdot 3^2\cdot 7^2\cdot 11 + 2^3\cdot 3^4\cdot 5\cdot 11 + 2^5\cdot 3^2\cdot 5\cdot 13 + 2^3\cdot 3^2\cdot 7\cdot 13 \\ & + 2^4\cdot 3^4\cdot 13 + 2^4\cdot 5^2\cdot 11\cdot 13 + 2^2\cdot 5\cdot 7\cdot 11\cdot 13 + 2^2\cdot 5\cdot 7\cdot 11\cdot 13 \\ & + 7^2\cdot 11\cdot 13) = 3617/(2\cdot 3\cdot 5\cdot 17)\,. \end{aligned}$$

By induction, we express the Bernoulli number B_k by

(25) $$B_k = \prod_{p<2k+2} p^{a(p,k)} \sum_{r=1}^{c(k)} \prod_{p<2k} p^{b(p,r,k)}\,.$$

Where the products are over the primes less than $2k+2$ and $2k$, respectively, $a(p,k)$ is an integer (possibly negative) and $b(p,r,k)$ is a non-negative integer. The number $c(k)$ of terms in the sum clearly possesses the recurrence

(26) $$c(k) = \sum_{r=1}^{[k/2]} c(r)c(k-r)\,,$$

with initial condition $c(1) = 1$. Kishore [12], [13] has used this technique to develop analogous structure theorems for Rayleigh functions [14], [15].

3. DIVISIBILITY STRUCTURE OF THE BERNOULLI NUMBERS

We first cite the well-known [16], [17]

Theorem 2. (Von Staudt-Clausen). If $B_k = P_k/Q_k$ are the Bernoulli numbers for $k = 1, 2, 3, \cdots$ and $(P_k, Q_k) = 1$, then

$$Q_k = \prod_{p-1 \mid 2k} p, \tag{27}$$

where the product is over all primes whose totients divide $2k$.

This theorem completely characterizes the Bernoulli denominators; hence, questions of divisibility center around the numerators P_k. A sufficient condition on divisors of P_k is given in the following [16, p. 261]

Theorem 3. If $p^{\omega} \mid 2k$, $p^{\omega+1} \nmid 2k$, $p - 1 \nmid 2k$, then $p^{\omega} \mid P_k$.

The proof of this theorem follows from a congruence of Voronoi

$$(a^{2k} - 1)P_k \equiv (-)^{k-1} 2k\, a^{2k-1} Q_k \sum_{s=1}^{N-1} s^{2k-1} [sa/N] \pmod{N}, \tag{28}$$

where $(a, N) = 1$ and N is any integer greater than one. Clearly if $p^{\omega} \mid 2k$, $(a^{2k} - 1)P_k \equiv 0 \pmod{p}$ and we may select a to be a primitive root g of p^{ω} (i.e., if $\omega = 1$, g always exists: if $\omega > 1$ and $g^{p-1} \not\equiv 1 \pmod{p^2}$, take $a = g$; if $g^{p-1} \equiv 1 \pmod{p^2}$, take $a = g + p$).

Equation (28) is a type of congruence used recently [18], [19] to investigate certain divisors of Bernoulli numerators. Specifically, those primes p such that

$$p \nmid P_1 P_2 P_3 \cdots P_{(p-3)/2} \tag{29}$$

are called regular primes and Kummer [20] proved that for these primes, Fermat's inequality, $x^p + y^p \neq z^p$, holds for all nonzero integers x, y and z. We list a number of congruences of the Voronoi type.

$$\sum_{p/6 < s < p/4} s^{2k-1} \equiv (2^{p-2k} - 1)(3^{p-2k} - 2^{p-2k} - 1)(-)^k B_k/4k \pmod{p} \tag{30}$$

with [16, p. 268], $p > 3$, $p - 1 \nmid 2k$

$$\sum_{p/6 < s < p/5} s^{2k-1} + \sum_{p/3 < s < 2p/5} s^{2k-1} \equiv (-)^k (6^{p-2k} - 5^{p-2k} - 2^{p-2k} + 1) B_k/4k \pmod{p} \tag{31}$$

with [19, p. 27], $p > 7$, $2k < p - 1$.

$$\sum_{p/6 < s < p/3} s^{2k-1} \equiv (-)^k (2^{p-2k-1} - 1)(3^{p-2k} - 1) B_k / 2k \pmod{p} \tag{32}$$

with [21], $p > 7$, $2k < p - 1$.

$$\sum_{r=1}^{(p-1)/2} (p - 2r)^{2k} \equiv p 2^{2k-1} B_k \pmod{p^3} \tag{33}$$

with [22], $2k \not\equiv 2 \pmod{(p - 1)}$.

$$b^{a(p-1)} (b^{p-1} - 1)^j \equiv 0 \pmod{p^{j-1}} \tag{34}$$

with [23], p an odd prime, $a > 0$, $j > 0$, $a + j < p - 1$.

From reflections on the divisibility properties of the binomial coefficients, it has been shown [24] that

$$2B_k \equiv 1 \pmod{2^{r+1}}, \quad \text{for } k > 1,\ 2^r \mid 2k,\ 2^{r+1} \nmid 2k . \tag{35}$$

Also [25],

$$2B_k \equiv 1 \pmod{4}, \quad k > 1 , \tag{36}$$

and [26],

$$B_k \equiv 1 - (1/p) \pmod{p^r}, \quad \text{for } p > 2,\ (p - 1)p^r \mid 2k,\ p^{r+1} \nmid 2k . \tag{37}$$

A more elaborate result [2] is

$$30B_{2k} \equiv 1 + 600 \binom{k - 1}{2} \pmod{27000} . \tag{38}$$

The last depends upon special identities such as

$$(e^x - 1)^{-1} - (e^{5x} - 1)^{-1} = \big(\cosh(x/2) + \cosh(3x/2)\big) \cosh(5x/2) .$$

4. APPROACHES TO CONJECTURE 1

Milnor [1, p. 966] asked whether or not

(39) $$8(2k)!/(2^{2k-1} - 1)B_k \not\equiv 0 \pmod 1 .$$

That this is true for $k > 2$ is clear by remarking [27] that $2^{2k-1} - 1$ possesses a primitive divisor q, such that $q \equiv 1 \pmod{2k-2}$.

In particular, $q > 2k + 1$ and q must occur in the denominator of the fraction in (39). We naturally ask whether or not a prime $q > 2k + 1$ always exists such that

$$q \mid 2^{2k-1} - 1 \quad \text{and} \quad q \nmid 2^{2r-1} - 1, \quad q \nmid 2^{2k-2r-1} - 1, \quad q \nmid B_r, \quad q \nmid B_{k-r} .$$

with $k/3 < r \le k/2$. This suggests

<u>Lemma 1.</u> If $q \mid 2^{2k-1} - 1$ is primitive and regular, then Conjecture 1 is true for k.

We consider $r = k/2$ or $(k-1)/2$, $k > 3$. Since $q > 2k + 1$ and $q \nmid B_i$ for $i < (q-1)/2$, $q \nmid B_r^2$, if k is even and $q \nmid B_r B_{k-r}$ if k is odd. Also [28], $q \nmid 2^j - 1$, $j < 2k - 1$. Another natural question is, since Fermat's Last Theorem is true for [29] primes of the form $2^a - 1$, are these numbers and their large factors also regular? Alas,

$$233 \mid B_{42}, \qquad 233 \mid 2^{29} - 1 .$$

As an example of the theorem, $k = 15$, $2k - 1 = 29$; $1103 \mid 2^{29} - 1$, yet 1103 is regular; the nearest irregular primes are 971 and 1061. Also $3391 \mid B_{1116}$, $3391 \mid B_{1267}$ and $3391 \mid 2^{113} - 1$, but $3391 \nmid B_{28}B_{29}$ so that irregular primes may be primitive and still satisfy conjecture 1. Similarly for $263 \mid 2^{131} - 1$ and $263 \mid B_{50}$. These remarks handle cases $k = 57, 66$. The number of primitive primes is infinite. so is the number of irregular primes [30]; Kummer conjectured that the number of regular primes is infinite. Present tables show that known regular primes are more numerous than irregular primes. The intersection of these primitive and regular prime sets, though nonempty, is unknown. It is interesting to note in this connection that

(40) $$2^{2k-1} - 1 = \sum_{r=1}^{k} \binom{2k-1}{2r-1} (2^{2k-2r-1} - 1)(2^{2r} - 1)B_r / r ,$$

which for $2k - 1$ prime is a relation between Mersenne [31] numbers and Bernoulli numbers. We might enjoy having $(2^{4k-1} - 1, B_k) = 1$, for the case of the $(8k-1)$-sphere; but

$$(2^{27} - 1, B_7) = (2^{111} - 1, B_{28}) = 2^3 - 1,$$

and a similar thing occurs whenever $3 \mid 4k - 1$, $7 \mid 2k$; likewise, if $5 \mid 4k - 1$, $31 \mid 2k$, e.g., $(2^{495} - 1, B_{124}) \ge 31$.

Another approach to (39) is to seek a large (greater than 2k) prime factor of B_k and to apply its existence to Conjecture 1. However, there does not appear to be in the literature any theorem (other than a direct calculation [32] proving the existence of a large prime divisor of B. Equation (25) suggests that if the $b(p, r, k)$ numbers behave appropriately, the sum in (25) would be the source of large factors; for the first few cases the sum has a number of small factors (i.e., equations (17)-(24)). A very general and related problem is whether or not sums of the type

(41) $$\sum_{r=1}^{c(k)} \prod_{p<2k} p^{\eta(p,r,k)}$$

with the function $\eta(p, r, k)$ behaving similarly to the $b(p, r, k)$ possess large factors. It is known 33 that for sums of type (41) where $\eta(p, r, k) \gg b(p, r, k)$ (inequality in a rough distribution sense of the density of primes being greater in one than the other) large factors arise. One must proceed with considerable care because of the copious factors [34] of a sum such as

(42) $$\sum \binom{n}{a_1, \cdots, a_k} \binom{n(k-1)}{n-a_1, \cdots, n-a_k} = \binom{nk}{n, \cdots, n} ,$$

where the sum is over the partitions

$$\sum_{i=1}^{k} a_i = n .$$

Rather than digging a prime out of P_k, we recognize the obvious

Lemma 2. For m,n arbitrary positive integers, such that $m/n < 1$, then there exists a prime p such that $p \mid n/(m,n)$ and $p \nmid m/(m,n)$.

We write for integers $r \in (k/3, k/2]$, $k > 3$,

(43) $$\binom{2k}{2r} (2^{2r-1} - 1)(2^{2k\ 2r-1} - 1)B_r B_{k-r} / (2^{2k-1} - 1)B_k$$

(44) $$= \binom{2k}{2r} (Q_k / Q_r Q_{k-r})(2^{2r-1} - 1)(2^{2k-2r-1} - 1)P_r P_{k-r} / (2^{2k-1} - 1)P_k$$

$$= \binom{2k}{2r} {}_{p<2k+2} \; p^{\theta(p,k)-\theta(p,r)-\theta(p,k-r)} (2^{2r-1} - 1)(2^{2k-2r-1} - 1)P_r P_{k-r}$$

(45) $$/M_k M_k' N_k N_k' ,$$

where

(46) $$\theta(p,k) = 1 \text{ if } (p-1) \mid 2k \text{ and zero otherwise}$$

with

(47) $$2^{2k-1} - 1 = M_k M_k' , \quad M_k = \prod_{p<2k} p^{\Psi(p,k)}, \quad M_k \text{ largest possible,}$$

and

$$P_k = N_k N_k', \quad N_k = \prod_{p<2k} p^{\varphi(p,k)}, \quad N_k \text{ largest possible.} \tag{48}$$

Therefore, we have the following

Lemma 3. If

$$M_k N_k < 0.25 \binom{2k}{2r} Q_k / Q_r Q_{k-r} \tag{49}$$

for some integer $r \in (k/3, k/2]$, then Conjecture 1 is true.

From (3),

$$B_r B_{k-r} / B_k = \binom{2k}{2r}^{-1} 2\zeta(2r)\zeta(2k - 2r)/\zeta(2k) < 4 \Big/ \binom{2k}{2r}. \tag{50}$$

In fact, [35], for k even,

$$\zeta^2(k)/\zeta(2k) = \sum_{n=1}^{\infty} 2^{\nu(n)} / n^k , \tag{51}$$

for $\nu(n)$ equal to the number of distinct prime factors of n.

By hypothesis

$$\begin{aligned} m/n &= (2^{2r-1} - 1)(2^{2k-2r-1} - 1) P_r P_{k-r} / M_k' N_k' \\ &< 4 M_k N_k \binom{2k}{2r}^{-1} Q_r Q_{k-r} / Q_k < 1 . \end{aligned} \tag{52}$$

But n has no prime factors less than $2k$ and hence none less than $2(k - r)$ (whether $2k + 1$ is prime or not, n has no factors less than $2k + 2$), so by Lemma 2 there exists some prime greater than $2k$, which provides a non-trivial bound for Conjecture 1. Also, if $2k - 1$ is prime, $M_k = 1$; in general, for say $n = 2k - 1$, an easily refined inequality is $M_k \le n2^{\varphi(n)+2^{0.09\nu(n)}}$ with φ Euler's totient function.

Since for relatively small k, discovery of a large prime divisor of P_k could require more than 10^{38} centuries with our present technology, Lemma 3 presents itself as a most opportune calculational device. Using this lemma we have shown Conjecture 3 to be true for integers $k \in (3, 265]$. The details of this calculation, which appear in the appended tables, materially suggest the truth of the hypothesis of Lemma 3. These calculations make use of congruences of type (28), which gives necessary conditions for all divisors of P_k, conditions which depend upon properties of the sum

$$\sum_{s=1}^{p^{\omega}-1} s^{2k-1}\left[sa/p^{\omega}\right], \qquad (\text{mod } p^{\omega}), \tag{53}$$

for a some primitive root of p (a complication can arise here because p = 3511, which satisfies $2^{p-1} \equiv 1 \pmod{p^2}$, has a Kummer irregularity of 2).

Of (53), the tables present empirical evidence, the most complete to date; the more valuable conceptual information in the form of an upper bound inequality on N_k, for example, would be welcome knowledge at this point.

REFERENCES

1. J. Milnor, "Differentiable Structures on Spheres," Amer. J. Math., Vol. 81 (1959), pp. 962-971.
2. J. S. Frame, "Bernoulli Numbers Modulo 27000," Amer. Math. Monthly, Vol. 68 (1961) pp. 87-95.
3. K. Knopp, Infinite Sequences and Series, Dover, p. 172.
4. E. T. Bell, "Exponential Numbers," Amer. Math. Monthly, Vol. 41 (1934), pp. 411-419.
5. P. A. MacMahon, Combinatory Analysis, Chelsea, pp. 3-4.
6. C. Jordan, Calculus of Finite Differences, Chelsea, p. 247.
7. M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, Nat. Bureau of Stds., pp. 823, 831-832.
8. R. Ayoub, An Introduction to the Analytic Theory of Numbers, Amer. Math. Soc. MS 10, pp. 134-205.
9. J. C. Adams, "Table of the First Sixty-Two Numbers of Bernoulli," J. für Reine und Angewandte Math., Vol. 85 (1878), pp. 269-272.
10. D. H. Lehmer, "Lacunary Recurrence Formulas for the Numbers of Bernoulli and Euler," Ann. of Math., 2nd Ser., Vol. 36 (1935), pp. 637-649.
11. H. T. Kuo, "A Recurrence Formula for $\zeta(2n)$," Bull. Amer. Math. Soc., Vol. 55 (1949) pp. 573-574.
12. N. Kishore, "A Structure of the Rayleigh Polynomial," Duke Math. J., Vol. 31 (1964) pp. 513-518.
13. N. Kishore, "A Representation of the Bernoulli Number B_n," Pacific J. Math., Vol. 14 (1964), pp. 1297-1304.
14. G. N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge, p. 502.
15. D. H. Lehmer, "Zeros of the Bessel Function $J_\nu(x)$," Math Tables and Other Aids to Computation, Vol. 1 (1943-45), pp. 405-407.
16. J. V. Uspensky and M. A. Heaslet, Elementary Number Theory, McGraw-Hill, pp. 257-258.
17. K. von Staudt, "Beweis eines Lehrsatzes, die Bernoullischen Zahlen betreffend," J. für Reine und Angewandte Math., Vol. 21 (1840), pp. 372-274.

18. D. H. Lehmer, Emma Lehmer, and H. S. Vandiver, "An Application of High-Speed Computing to Fermat's Last Theorem," Proc. Nat. Acad. Sci., USA, Vol. 40 (1954), pp. 25-33.

19. J. L. Selfridge, C. A. Nichol, and H. S. Vindiver, "Proof of Fermat's Last Theorem for all Prime Exponents less than 4002," Proc. Nat. Acad. Sci., USA, Vol. 41 (1955), pp. 970-973.

20. E. E. Kummer, J. für Reine und Angewandte Math., Vol. 40 (1850), pp. 93-138.

21. E. T. Stafford and H. S. Vindiver, "Determination of Some Properly Irregular Cyclotomic Fields," Proc. Nat. Acad. Sci., USA, Vol. 16 (1930), pp. 139-150.

22. Emma Lehmer, "On Congruences Involving Bernoulli Numbers and the Quotients of Fermat and Wilson," Annals Math., 2nd Ser., Vol. 39 (1938), pp. 350-360.

23. H. S. Vandiver, "Certain Congruences Involving the Bernoulli Numbers," Duke Math. J., Vol. 5 (1939), pp. 548-551.

24. L. Carlitz, "A Note on the Staudt-Clausen Theorem," Amer. Math. Monthly, Vol. 66 (1957), pp. 19-21.

25. L. Carlitz, "A Property of the Bernoulli Numbers," Amer. Math. Monthly, Vol. 66 (1959), pp. 714-715.

26. L. Carlitz, "Some Congruences for the Bernoulli Numbers," Amer. J. Math., Vol. 75 (1953), pp. 163-172.

27. G. D. Birkhoff and H. S. Vandiver, "On the Integral Divisors of $a^n - b^n$," Annals Math. Vol. 5 (1903), pp. 173-180.

28. P. Erdős, "On the Converse of Fermat's Theorem," Amer. Math. Monthly, Vol. 56 (1949), pp. 623-624.

29. H. S. Vandiver, Amer. Math. Soc., Vol. 15 (1914), p. 202. Transactions

30. K. W. L. Jensen, Nyt Tidsskrift for Mathematik, Afdeling B, Vol. 82 (1915), (Math. Reviews, pp. 27-2475.

31. R. M. Robinson, "Mersenne and Fermat Numbers," Proc. Amer. Math. Soc., Vol. 5 (1954), pp. 842-846.

32. N. G. W. H. Beeger, "Report on Some Calculations of Prime Numbers," Nieuw Arch. Wiskde., Vol. 20 (1939), pp. 48-50, Math. Reviews, Vol. 1 (1940), p. 65.

33. L. Carlitz, "A Sequence of Integers Related to the Bessel Functions," Proc. Amer. Math. Soc., Vol. 14 (1963), pp. 1-9.

34. L. Carlitz, "Sums of Products of Multinomial Coefficients," Elem. Math., Vol. 18 (1963), pp. 37-39, Math. Reviews, pp. 27-56.

35. E. C. Titchmarsh, The Theory of the Riemann Zeta-Function, Oxford, p. 5.