ENUMERATION OF TWO-LINE ARRAYS

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1. We consider the enumeration of two-line arrays of positive integers

$$\begin{array}{|cccc|} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{array} \tag{1.1}$$

subject to certain conditions. We assume first that

$$\max(a_i, b_i) \le \min(a_{i+1}, b_{i+1}) \qquad (1 \le i < n) \tag{1.2}$$

and

$$\max(a_i, b_i) \le i \qquad (1 \le i \le n)\,. \tag{1.3}$$

Let $f(n,k)$ denote the number of arrays (1.1) satisfying (1.2) and (1.3) and in addition

$$a_n = b_n = k\,; \tag{1.4}$$

let $g(n,k)$ denote the number of arrays (1.1) satisfying (1.2) and (1.3) and

$$\max(a_n, b_n) = k\,. \tag{1.5}$$

Also put

$$f(n) = f(n,n), \qquad g(n) = g(n,n)\,. \tag{1.6}$$

Next let $h(n,k)$ denote the number of arrays (1.1) that satisfy the conditions

$$1 = b_1 = a_1 \le b_2 \le a_2 \le \cdots \le b_n \le a_n = k \tag{1.7}$$

and

$$a_i \le i \qquad (1 \le i \le n)\,. \tag{1.8}$$

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Also put

$$(1.9) \qquad h(n) = \sum_{k=1}^{n} h(n,k) .$$

We shall determine the enumerants f, g, h explicitly. In particular, we show that

$$(1.10) \qquad f(n+1) = \frac{1}{n}\sum_{t=1}^{n}\binom{n}{t}\binom{2n+t}{t-1} ,$$

$$(1.11) \qquad h(n) = \frac{1}{n}\binom{3n}{n-1} .$$

Note that $f(n+1)$ is the total number of arrays satisfying (1.2) and (1.3), while $h(n)$ is the total number of arrays satisfying (1.8) and

$$(1.7)' \qquad 1 = b_1 = a_1 \le b_2 \le a_2 \le \cdots \le b_n \le a_n .$$

The conditions (1.2), (1.3) are suggested by one formulation of the ballot problem (for references see [2]). On the other hand, (1.2) has also occurred in a problem in multipartite partitions [1], [4].

2. To begin with, we consider the functions $f(n,k)$, $g(n,k)$. We state some preliminary results.

$$(2.1) \qquad f(n+1,k) = \sum_{j=1}^{k}(2k-2j+1)f(n,j) \qquad (k \le n) ,$$

$$(2.2) \qquad f(n+1,k) = \sum_{j=1}^{k} g(n,j) \qquad (k \le n+1) ,$$

$$(2.3) \qquad g(n+1,k) = \sum_{j=1}^{k}(2k-2j+1)g(n,j) \qquad (k \le n) ,$$

$$(2.4) \qquad g(n,k) = f(n,k) + 2\sum_{j=1}^{k-1} f(n,j) \qquad (k \le n+1) ,$$

$$g(k+1,\ k) = \sum_{j=1}^{k} g(j,j)g(k-j+1,\ k-j+1)\ . \tag{2.5}$$

To prove (2.1), consider the array

$$\boxed{\begin{matrix} a_1 & a_2 & \cdots & a_n & k \\ b_1 & b_2 & \cdots & b_n & k \end{matrix}}$$

where

$$\max(a_i,\ b_i) \le \min(a_{i+1},\ b_{i+1}) \qquad (1 \le i \le n)\ ,$$
$$\max(a_n,\ b_n) \le k\ ,$$
$$\max(a_i,\ b_i) \le i \qquad (1 \le i \le n)\ .$$

Let $j = \min(a_n,\ b_n)$. For fixed $j \le k$, we can pick a_n, b_n in $2k - 2j + 1$ ways. This evidently implies (2.1).

Equation (2.2) is an immediate consequence of the definitions. The proof of (2.3) is similar to the proof of (2.1). We consider the array

$$\boxed{\begin{matrix} a_1 & a_2 & \cdots & a_n & a_{n+1} \\ b_1 & b_2 & \cdots & b_n & b_{n+1} \end{matrix}}$$

where now

$$\max(a_n,\ b_n) = j, \qquad \max(a_{n+1},\ b_{n+1}) = k\ .$$

For fixed j,k, we can pick a_{n+1}, b_{n+1} in $2k - 2j + 1$ ways. This yields (2.3).

As for (2.4), it is only necessary to observe that corresponding to the array

$$\boxed{\begin{matrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{matrix}}$$

where $\max(a_n,\ b_n) = k$, we have the set of arrays

$$\boxed{\begin{matrix} a_1 & a_2 & \cdots & a_{n-1} & j \\ b_1 & b_2 & \cdots & b_{n-1} & j \end{matrix}}$$

where $j = \min(a_n,\ b_n)$.

To prove (2.5), consider

$$(2.6) \qquad \underbrace{\begin{array}{|ccc}1 & \cdots & j \\ 1 & \cdots & \cdot\end{array}}_{j}\underbrace{\begin{array}{|ccc|}j & \cdots & k \\ j & \cdots & \cdot\end{array}}_{k-j+1}$$

Since

$$\max(a_1, b_1) = 1, \qquad \max(a_{k+1}, b_{k+1}) = k,$$

there is a least j such that

$$\max(a_j, b_j) = \max(a_{j+1}, b_{j+1}).$$

Thus $a_{j+1} = b_{j+1} = j$. Subtracting $j - 1$ from each element in the right-hand sub-array of (2.6), we get (2.5).

A more general result is

$$(2.7) \qquad g(n+k,k) = \sum_{j=1}^{k} g(j,j)g(n+k-j, k-j+1) \qquad (n \geq 1).$$

To prove (2.7), we consider the array

$$\underbrace{\begin{array}{|ccc}1 & \cdots & j \\ 1 & \cdots & \cdot\end{array}}_{j}\underbrace{\begin{array}{|ccc|}j & \cdots & k \\ j & \cdots & \cdot\end{array}}_{n+k-j}$$

and pick j as in the proof of (2.5).

Next we have

$$(2.8) \qquad f(k+1,k) = \sum_{j=1}^{k} g(j,j)f(k-j+1, k-j+1)$$

and

$$(2.9) \qquad f(n+k,k) = \sum_{j=1}^{k} g(j,j)f(n+k-j, k-j+1) \qquad (n \geq 1).$$

The proof of these formulas is similar to the proof of (2.5) and (2.7).

3. Put

(3.1) $$f(n) = f(n,n) = f(n, n-1), \qquad g(n) = g(n,n),$$

(3.2) $$F(x,y) = \sum_{n=1}^{\infty} \sum_{k=1}^{n} f(n,k)x^n y^k ,$$

(3.3) $$G(x,y) = \sum_{n=1}^{\infty} \sum_{k=1}^{n} g(n,k)x^n y^k ,$$

(3.4) $$F(x) = \sum_{n=1}^{\infty} f(n)\, x^n ,$$

(3.5) $$G(x) = \sum_{n=1}^{\infty} g(n)\, x^n .$$

We rewrite (2.8) in the form

(3.6) $$f(k+1) = \sum_{j=1}^{k} g(j)f(k-j+1) .$$

Then by (3.4),

$$\begin{aligned} F(x) &= x + \sum_{k=1}^{\infty} f(k+1)x^{k+1} \\ &= x + \sum_{k=1}^{\infty} x^{k+1} \sum_{j=1}^{k} g(j)f(k-j+1) \\ &= x + \sum_{j=1}^{\infty} g(j)x^j \sum_{k=1}^{\infty} f(k)x^k , \end{aligned}$$

so that

(3.7) $$F(x) = x + F(x)G(x) .$$

Next, by (2.7),

$$\sum_{k=1}^{\infty} g(n+k,k)x^{n+k} = \sum_{k=1}^{\infty} x^{n+k} \sum_{j=1}^{\infty} g(j,j)g(n+k-j,k-j+1) = \sum_{j=1}^{\infty} g(j,j)x^j \sum_{k=1}^{\infty} g(n+k-1,k)x^{n+k-1} .$$

Hence, if we put

$$G_n(x) = \sum_{k=1}^{\infty} g(n + k - 1, k)\, x^{n+k-1} \qquad (n \geq 1), \tag{3.8}$$

we have

$$G_{n+1}(x) = G(x)G_n(x) \qquad (n \geq 1). \tag{3.9}$$

Since

$$G_1(x) = \sum_{k=1}^{\infty} g(k,k)x^k = G(x),$$

it follows that

$$G_n(x) = G^n(x). \tag{3.10}$$

Next consider

$$\begin{aligned} G(x,y) &= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} g(n,k)x^n y^k \\ &= \sum_{j,k=1}^{\infty} g(j + k - 1, k)\, x^{j+k-1} y^k \\ &= \sum_{j=1}^{\infty} y^{-j+1} \sum_{k=1}^{\infty} g(j + k - 1, k)(xy)^{j+k-1} \\ &= \sum_{j=1}^{\infty} y^{-j+1} G^j(xy), \end{aligned}$$

by (3.10). Therefore

$$G(x,y) = \frac{G(xy)}{1 - y^{-1}G(xy)}. \tag{3.11}$$

On the other hand, by (2.3),

$$\begin{aligned} G(x,y) &= xy + x\sum_{n=1}^{\infty}\sum_{k=1}^{n+1} g(n + 1, k)\, x^n y^k \\ &= xy + \sum_{n=1}^{\infty} g(n + 1, n + 1)(xy)^{n+1} + x\sum_{n=1}^{\infty}\sum_{k=1}^{n}\sum_{j=1}^{k} (2k-2j+1)g(n,j)x^n y^k = \end{aligned}$$

$$= G(xy) + x \sum_{n=1}^{\infty} \sum_{j=1}^{n} g(n,j)x^n y^j \sum_{k=0}^{n-j} (2k+1)y^k$$

$$= G(xy) + x \sum_{n,j=1}^{\infty} g(n+j-1, j)x^{n+j-1} y^j \sum_{k=0}^{n-1} (2k+1)y^k$$

$$= G(xy) + x \sum_{n=1}^{\infty} y^{-n+1} \sum_{k=0}^{n-1} (2k+1)y^k \sum_{j=1}^{\infty} g(n+j-1, j)(xy)^{n+j-1}$$

$$= G(xy) + x \sum_{n=1}^{\infty} y^{-n+1} \sum_{k=0}^{n-1} (2k+1)y^k \cdot G^n(xy)$$

$$= G(xy) + x \sum_{n,k=0}^{\infty} (2k+1)y^{-n} G^{n+k+1}(xy)$$

$$= G(xy) + \frac{xG(xy)}{1 - y^{-1}G(xy)} \sum_{k=0}^{\infty} (2k+1)G^k(xy) .$$

Since

$$\sum_{k=0}^{\infty} (2k+1)z^k = \frac{1+z}{(1-z)^2} ,$$

it follows that

$$G(x,y) = G(xy) + \frac{xG(xy)}{1 - y^{-1}G(xy)} \frac{1 + G(xy)}{(1 - G(xy))^2} \tag{3.12}$$

Comparing (3.12) with (3.11), we get

$$\frac{1}{1 - y^{-1}G(xy)} = 1 + \frac{x}{1 - y^{-1}G(xy)} \frac{1 + G(xy)}{(1 - G(xy))^2} .$$

For $y = 1$, this reduces to

$$\frac{1}{1 - G(x)} = 1 + \frac{x}{1 - G(x)} \frac{1 + G(x)}{(1 - G(x))^2}$$

Therefore

$$G(x)(1 - G(x))^2 = x(1 + G(x)) . \tag{3.13}$$

Now consider the equation

$$z(1-z)^2 = x(1+z) , \tag{3.14}$$

where $z = 0$ when $x = 0$. By [3, p. 125], the equation

$$w = \frac{z}{\phi(z)} \qquad (\phi(0) = 1), \tag{3.15}$$

where $\phi(z)$ is analytic in the neighborhood of $z = 0$, has the solution

$$z = \sum_{n=1}^{\infty} \frac{w^n}{n!} \left[\frac{d^{n-1}\phi^n(x)}{dx^{n-1}}\right]_{x=0} . \tag{3.16}$$

Later we shall require the more general result:

$$f(z) = f(0) + \sum_{n=1}^{\infty} \frac{w^n}{n!} \left[\frac{d^{n-1}}{dx^{n-1}} f'(x)\phi^n(x)\right]_{x=0} . \tag{3.17}$$

If we take

$$\phi(x) = (1 + x)(1 - x)^{-2},$$

then

$$\phi^n(x) = \sum_{s=0}^{\infty} \binom{n}{s} x^s \sum_{t=0}^{\infty} \binom{2n + t - 1}{t} x^t$$

$$= \sum_{m=0}^{\infty} x^m \sum_{s+t=m} \binom{n}{s}\binom{2n + t - 1}{t} ,$$

so that

$$\left[\frac{d^{n-1}}{dx^{n-1}} \phi^n(x)\right]_{x=0} = (n - 1)! \sum_{t=0}^{n-1} \binom{n}{t + 1}\binom{2n + t - 1}{t} .$$

Therefore, by (3.13) and (3.16), we get

$$G(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \sum_{t=0}^{n-1} \binom{n}{t + 1}\binom{2n + t - 1}{t} . \tag{3.18}$$

Thus

$$g(n) = \frac{1}{n} \sum_{t=0}^{n-1} \binom{n}{t + 1}\binom{2n + t - 1}{t} . \tag{3.19}$$

4. In the next place, by (3.7),

$$F(x) = \frac{x}{1 - G(x)} . \tag{4.1}$$

Then, making use of (3.17),

$$\frac{F(x)}{x} = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} \left[\frac{d^{n-1}}{dx^{n-1}} \frac{(1 + x)^n}{(1 - x)^{2n+2}}\right]_{x=0} . \tag{4.2}$$

It is easily verified that

$$\left[\frac{d^{n-1}}{dx^{n-1}} \frac{(1 + x)^n}{(1 - x)^{2n+2}}\right]_{x=0} = (n - 1)! \sum_{t=0}^{n-1} \binom{n}{t + 1}\binom{2n + t + 1}{t} ,$$

so that (4.2) yields

$$F(x) = x + \sum_{n=1}^{\infty} \frac{x^{n+1}}{n} \sum_{t=0}^{n-1} \binom{n}{t + 1}\binom{2n + t + 1}{t} . \tag{4.3}$$

Hence

$$f(n + 1) = \frac{1}{n} \sum_{t=0}^{n-1} \binom{n}{t + 1}\binom{2n + t + 1}{t} . \tag{4.4}$$

To determine $g(n,k)$ we use (3.10), that is

$$\sum_{k=1}^{\infty} g(j + k - 1, k)x^{j+k-1} = G^j(x) . \tag{4.5}$$

Taking $f(z) = z^j$ in (3.17), we get

$$G^j(x) = j \sum_{n=j}^{\infty} \frac{x^n}{n!} \left[\frac{d^{n-1}}{dx^{n-1}} \left(x^{j-1} \frac{(1 + x)^n}{(1 - x)^{2n}} \right)\right]_{x=0} .$$

Since

$$\frac{d^{n-1}}{dx^{n-1}} \left(x^{j-1} \frac{(1 + x)^n}{(1 - x)^{2n}} \right) = (n - 1)! \sum_{t=0}^{n-j} \binom{n}{j + t}\binom{2n + t - 1}{t} ,$$

it follows that

$$G^j(x) = j \sum_{n=j}^{\infty} \frac{x^n}{n} \sum_{t=0}^{n-j} \binom{n}{j + t}\binom{2n + t - 1}{t} . \tag{4.6}$$

Hence, by (4.5),

$$(4.7)\qquad g(n,\ n-j+1) = \frac{j}{n}\sum_{t=0}^{n-j}\binom{n}{j+t}\binom{2n+t-1}{t} .$$

Next if we put

$$(4.8)\qquad F_n(x) = \sum_{k=1}^{\infty} f(n+k-1,\ k)x^{n+k-1} \qquad (n \geq 1),$$

it follows from (2.9) that

$$F_{n+1}(x) = \sum_{k=1}^{\infty} f(n+k,\ k)x^{n+k}$$

$$= \sum_{k=1}^{\infty} x^{n+k}\sum_{j=1}^{k} g(j)f(n+k-j,\ k-j+1)$$

$$= \sum_{j=1}^{\infty} g(j)x^j \sum_{k=1}^{\infty} f(n+k-1,\ k)x^{n+k-1} ,$$

so that

$$(4.9)\qquad F_{n+1}(x) = F_n(x)G(x) \qquad (n \geq 1) .$$

Hence, by (4.9) and (4.1),

$$(4.10)\qquad F_{n+1}(x) = F(x)G^n(x) = \frac{xG^n(x)}{1-G(x)} \qquad (n \geq 0) .$$

We now apply (3.17) with

$$f(x) = \frac{x^j}{1-x} .$$

Since

$$f'(x) = \frac{jx^{j-1}}{1-x} + \frac{x^j}{(1-x)^2} ,$$

we get

$$\frac{1}{x} F_{j+1}(x) = f(0) + \sum_{n=j}^{\infty} \frac{x^n}{n}\left\{ j\sum_{t=0}^{n-j}\binom{n}{j+t}\binom{2n+t}{t} + \sum_{t=0}^{n-j-1}\binom{n}{j+t+1}\binom{2n+t+1}{t}\right\} .$$

It follows that

$$f(n+1,\ n-j+1) = \frac{1}{n}\left\{ j\sum_{t=0}^{n-j}\binom{n}{j+t}\binom{2n+t}{t} + \sum_{t=0}^{n-j-1}\binom{n}{j+t+1}\binom{2n+t+1}{t}\right\} ,$$

or if we prefer

(4.11) $$f(n+1, k+1) = \frac{1}{n}\left\{(n-k)\sum_{t=0}^{k}\binom{n}{k-t}\binom{2n+t}{t} + \sum_{t=1}^{k}\binom{n}{k-t}\binom{2n+t}{t-1}\right\} .$$

Note that, when $n = k$, (4.11) becomes

$$f(k+1) = f(k+1, k+1) = \frac{1}{k}\sum_{t=1}^{k}\binom{k}{t}\binom{2k+t}{t-1} ,$$

in agreement with (4.4).

5. The total number of arrays

(5.1) $$\begin{array}{|cccc|} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{array}$$

such that

(5.2) $$\begin{cases} \max(a_i, b_i) \le \min(a_{i+1}, b_{i+1}) & (1 \le i < n) \\ \max(a_i, b_i) \le i & (1 \le i \le n) , \end{cases}$$

is equal to

(5.3) $$\sum_{j=1}^{n} g(n,j) = f(n+1, n) = f(n+1) = \frac{1}{n}\sum_{t=0}^{n-1}\binom{n}{t+1}\binom{2n+t+1}{t} .$$

Similarly the total number of arrays (5.1) satisfying (5.2) and $a_n = b_n$ is equal to

(5.4) $$\sum_{k=1}^{n} f(n,k) = \frac{1}{2}\left(g(n,n) + f(n,n)\right) .$$

The numbers $f(n,k)$, $g(n,k)$ can be computed by means of the recurrences (2.1), (2.3). Checks are furnished by (2.2) and (2.4).

$f(n,k)$:

n \ k	1	2	3	4	5	6	7
1	1						
2	1	1					
3	1	4	4				
4	1	7	21	21			
5	1	10	47	126	126		
6	1	13	82	324	818	818	
7	1	16	126	642	2300	5594	5594

g(n,k) :

n \ k	1	2	3	4	5	6
1	1					
2	1	3				
3	1	6	14			
4	1	9	37	79		
5	1	12	69	242	494	
6	1	15	110	516	1658	3294

6. We turn now to $h(n,k)$, the number of arrays

$$\begin{array}{|cccc|} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{array}$$

such that

(6.1) $$1 = b_1 = a_1 \le b_2 \le a_2 \le \cdots \le b_n \le a_n = k$$

and

(6.2) $$a_i \le i \qquad (1 \le i \le n) .$$

It is clear from the scheme

$$\begin{array}{|ccccc|} a_1 & a_2 & \cdots & a_{n-1} & k \\ b_1 & b_2 & \cdots & b_{n-1} & j \end{array}$$

that

$$h(n,k) = \sum_{j=1}^{k} \sum_{s=1}^{j} h(n-1, s) .$$

This yields the recurrence

(6.3) $$h(n,k) = \sum_{s=1}^{k} (k - s + 1)h(n - 1, s) \qquad (1 \le k \le n) .$$

When $k = n$, it is understood that (6.3) becomes

(6.4) $$h(n,n) = \sum_{s=1}^{n-1} (n - s + 1)h(n - 1, s) .$$

It is clear from the definition that

(6.5) $$h(n) = \sum_{k=1}^{n} h(n,k)$$

is equal to the total number of arrays that satisfy (6.2) and

$$1 = b_1 = a_1 \le b_2 \le a_2 \le \cdots \le b_n \le a_n \tag{6.6}$$

The first few values of $h(n,k)$ can be computed by means of (6.2):

$h(n,k)$:

n \ k	1	2	3	4	5	h(n)
1	1					1
2	1	2				3
3	1	4	7			12
4	1	6	18	30		55
5	1	8	33	88	143	273

The numbers in the right-hand column are obtained by summing in the rows. Thus the entries are $h(n)$ as defined by (6.5).

We shall now show that

$$h(k+1, k) = \sum_{j=1}^{k} h(j,j)h(k-j+1, k-j+1) . \tag{6.7}$$

Proof. Consider the scheme

$$\underbrace{\begin{array}{|ccc|}\hline 1 & \cdots & j \\ 1 & \cdots & \cdot \\ \hline\end{array}}_{j}\underbrace{\begin{array}{ccc|}\hline j & \cdots & k \\ j & \cdots & \cdot \\ \hline\end{array}}_{k-j+1}$$

We choose j as the least positive integer such that

$$a_{j+1} = b_{j+1} = a_j .$$

Such an integer exists because $a_{k+1} = k$. Subtracting $j - 1$ from each element in the right-hand sub-array, we get (6.7).

Next we have

$$h(n+k, k) = \sum_{j=1}^{k} h(j,j)h(n+k-j, k-j+1) \qquad (n \ge 1) . \tag{6.8}$$

To prove (6.8), consider the scheme

$$\underbrace{\begin{array}{|ccc|}\hline 1 & \cdots & j \\ 1 & \cdots & \cdot \\ \hline\end{array}}_{j}\underbrace{\begin{array}{ccc|}\hline j & \cdots & k \\ j & \cdots & \cdot \\ \hline\end{array}}_{n+k-j}$$

We choose j as above. The rest of the proof is the same.

Now put

$$H(x,y) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} x^n y^k \, , \tag{6.9}$$

$$H(x) = \sum_{k=1}^{\infty} h(k,k)x^k = H_1(x) \, , \tag{6.10}$$

$$H_n(x) = \sum_{k=1}^{\infty} h(n+k-1, k)\, x^{n+k-1} \qquad (n \geq 1) \, . \tag{6.11}$$

Then, by (6.8),

$$\begin{aligned} H_{n+1}(x) &= \sum_{k=1}^{\infty} h(n+k, k)x^{n+k} \\ &= \sum_{k=1}^{\infty} x^{n+k} \sum_{j=1}^{k} h(j,j)h(n+k-j, k-j+1) \\ &= \sum_{j=1}^{\infty} h(j,j)x^j \sum_{k=1}^{\infty} h(n+k-1, k)x^{n+k-1} \, , \end{aligned}$$

so that

$$H_{n+1}(x) = H(x)H_n(x) \qquad (n \geq 1) \, . \tag{6.12}$$

Therefore

$$H_n(x) = H^n(x) \qquad (n \geq 1) \, . \tag{6.13}$$

In the next place

$$\begin{aligned} H(x,y) &= \sum_{n=1}^{\infty} \sum_{k=1}^{n} h(n,k)x^n y^k \\ &= \sum_{j,k=1}^{\infty} h(j+k-1, k)x^{j+k-1} y^k \\ &= \sum_{j=1}^{\infty} y^{-j+1} \sum_{k=1}^{\infty} h(j+k-1, k)(xy)^{j+k-1} \\ &= \sum_{j=1}^{\infty} y^{-j+1} H_j(xy) \, . \end{aligned}$$

Thus, by (6.13),

$$H(x,y) = \frac{H(xy)}{1 - y^{-1}H(xy)} \quad . \tag{6.14}$$

On the other hand, by (6.3),

$$\begin{aligned}
H(x,y) &= xy + x\sum_{n=1}^{\infty}\sum_{k=1}^{n+1} h(n+1,k)x^n y^k \\
&= xy + \sum_{n=1}^{\infty} h(n+1, n+1)(xy)^{n+1} + x\sum_{n=1}^{\infty}\sum_{k=1}^{n} h(n+1,k)x^n y^k \\
&= H(xy) + x\sum_{n=1}^{\infty}\sum_{k=1}^{n} x^n y^k \sum_{j=1}^{k} (k-j+1)h(n,j) \\
&= H(xy) + x\sum_{n=1}^{\infty}\sum_{j=1}^{n} h(n,j)x^n y^j \sum_{k=0}^{n-j} (k+1)y^k \\
&= H(xy) + x\sum_{n,j=1}^{\infty} h(n+j-1,j)x^{n+j-1} y^j \sum_{k=0}^{n-1} (k+1)y^k \\
&= H(xy) + x\sum_{n=1}^{\infty} y^{-n+1} \sum_{k=0}^{n-1} (k+1)y^k \sum_{j=1}^{\infty} h(n+j-1,j)(xy)^{n+j-1} \\
&= H(xy) + x\sum_{n=1}^{\infty} y^{-n+1} \sum_{k=0}^{n-1} (k+1)y^k H^n(xy) \\
&= H(xy) + x\sum_{n,k=0}^{\infty} (k+1)y^{-n} H^{n+k+1}(xy) \\
&= H(xy) + xH(xy)\sum_{k=0}^{\infty} (k+1)H^k(xy) \sum_{n=0}^{\infty} y^{-n} H^n(xy)
\end{aligned}$$

and therefore

$$H(x,y) = H(xy) + \frac{xH(xy)}{\left(1 - y^{-1}H(xy)\right)\left(1 - H(xy)\right)^2} \quad . \tag{6.15}$$

We now compare (6.15) with (6.14) and take $y = 1$. This yields

$$\frac{1}{1 - H(x)} = 1 + \frac{x}{\left(1 - H(x)\right)^3}$$

which reduces to

$$H(x)\left(1 - H(x)\right)^2 = x . \tag{6.16}$$

Applying (3.16), we get

$$H(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!} \left[\frac{d^{n-1}}{dt^{n-1}} (1 - t)^{-2n}\right]_{t=0} .$$

Since

$$\left[\frac{d^{n-1}}{dt^{n-1}} (1 - t)^{-2n}\right]_{t=0} = (n - 1)! \binom{3n - 2}{n - 1} ,$$

we have

$$H(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \binom{3n - 2}{n - 1} . \tag{6.17}$$

Applying (3.17), we get

$$H^j(x) = j \sum_{n=j}^{\infty} \frac{x^n}{n!} \left[\frac{d^{n-1}}{dt^{n-1}} \left(t^{j-1}(1 - t)^{-2n}\right)\right]_{t=0} \qquad (j \geq 1) .$$

This reduces to

$$H^j(x) = j \sum_{n=j}^{\infty} \frac{x^n}{n} \binom{3n - j - 1}{n - j} \qquad (j \geq 1) . \tag{6.18}$$

It follows from (6.11), (6.13) and (6.18) that

$$h(n, n - j + 1) = \frac{j}{n} \binom{3n - j - 1}{n - j} \qquad (1 \leq j \leq n) . \tag{6.19}$$

In particular, for $j = 1$, we have

$$h(n,n) = \frac{1}{n} \binom{3n - 2}{n - 1} . \tag{6.20}$$

We shall now compute

$$h(n) = \sum_{k=1}^{n} h(n,k) = \sum_{j=1}^{n} h(n, n - j + 1) . \tag{6.21}$$

By (6.19) and (6.21),

$$\begin{aligned} h(n) &= \sum_{j=1}^{n} \frac{j}{n}\binom{3n-j-1}{n-j} = \frac{1}{n}\sum_{j=0}^{n-1}(n-j)\binom{2n+j-1}{j} \\ &= \sum_{j=0}^{n-1}\binom{2n+j-1}{j} - \frac{1}{n}\sum_{j=1}^{n-1} j\binom{2n+j-1}{j} \\ &= \sum_{j=0}^{n-1}\binom{2n+j-1}{j} - 2\sum_{j=1}^{n-1}\binom{2n+j-1}{j-1} \\ &= \sum_{j=0}^{n-1}\binom{2n+j-1}{2n-1} - 2\sum_{j=0}^{n-2}\binom{2n+j}{2n} \\ &= \binom{3n-1}{2n} - \binom{3n-1}{2n+1} . \end{aligned}$$

This reduces to

$$h(n) = \frac{1}{n}\binom{3n}{n-1} . \tag{6.22}$$

7. By (5.3),

$$f(n+1) = \frac{1}{n}\sum_{t=1}^{n}\binom{n}{t}\binom{2n+t}{t-1} \tag{7.1}$$

enumerates the number of arrays that satisfy (5.1) and (5.2).

Consider the quantity

$$U(n,t) = \frac{1}{n}\binom{n}{t}\binom{2n+t}{t-1} \qquad (1 \le t \le n) . \tag{7.2}$$

Clearly $nU(n,t)$ is an integer Moreover

$$U(n,t) = \frac{1}{t}\binom{n-1}{t-1}\binom{2n+t}{t-1} = \frac{1}{2n+1}\binom{n-1}{t-1}\binom{2n+1}{t} ,$$

so that $(2n+1)U(n,t)$ is also an integer. Since both $nU(n,t)$ and $(2n+1)U(n,t)$ are integers, it follows that $U(n,t)$ is itself an integer. The question then arises whether $U(n,t)$ can be given a simple combinatorial interpretation. In the special case $t = n$, we have, by (6.22)

$$U(n,n) = h(n) ; \tag{7.3}$$

however the general case remains open.

A curious relation between $G(x)$ and $H(x)$ is implied by (3.13) and (6.16):

$$G(x)\big(1 - G(x)\big)^2 = x\big(1 + G(x)\big) , \tag{7.4}$$

(7.5) $$H(x)\bigl(1 - H(x)\bigr)^2 = x .$$

Since the equation

(7.6) $$z(1 - z)^2 = u, \qquad z(0) = 0$$

has the unique solution $z = H(u)$, it follows from (7.4) and (7.5) that

(7.7) $$H\Bigl(x\bigl(1 + G(x)\bigr)\Bigr) = G(x) .$$

By (3.8), (3.10), (6.11) and (6.13), we have

$$\begin{aligned} H\Bigl(x\bigl(1 + G(x)\bigr)\Bigr) &= \sum_{k=1}^{\infty} h(k,k)x^k \bigl(1 + G(x)\bigr)^k \\ &= \sum_{k=1}^{\infty} h(k,k)x^k \sum_{j=0}^{k} \binom{k}{j} G^j(x) \\ &= H(x) + \sum_{k=1}^{\infty} h(k,k)x^k \sum_{j=1}^{k} \binom{k}{j} \sum_{s=1} g(j + s - 1, s)x^{j+s-1} \\ &= H(x) + \sum_{n=2}^{\infty} x^n \sum_{j+k+s-1=n} \binom{k}{j} h(k,k)g(j + s - 1, s) . \end{aligned}$$

Thus (7.7) yields

(7.8) $$g(n) = h(n,n) + \sum_{j+k \le n} \binom{k}{j} h(k,k)g(n - k, n - j - k + 1) .$$

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